Boundedly Rational Monopoly with Continuously Distributed Single Time Delay*

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Abstract

In this paper we examine dynamics of a boundedly rational monopoly with continuously distributed time delay. Constructing the gradient dynamic system where the output change is proportional to the expected profit and the expected demand is formed based on past data with various shapes of the weighting function, three main results are analytically as well as numerically demonstrated: (1) the stability region depends on the shape of the weighting function and converges to the stability region of the fixed time delay when the shape parameter goes to infinity; (2) delay has a threshold value below which stability is preserved and above which it is lost; (3) the equilibrium point bifurcates to a limit cycle through Hopf bifurcation when it loses stability.

Key words: Boundedly rational, Continuously distributed time delay, Hopf bifurcation, Limit cycle, Gradient dynamics

JEL Classification: C62, C63, D21, D42

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1 Introduction

In most traditional models, instantaneous and complete information has been assumed. Recent research on the dynamic behavior of the economic agents, however, emphasizes on their bounded rationality that arises when the agents have only limited information in making their decisions. In this paper, we build a dynamic monopoly model that accounts for delay information and then explore analytically and numerically the delay effects upon local and global dynamic behavior of a monopolist or monopolistic firm.

In real economies there is always a delay in obtaining information and implementing decisions. In the existing literature two types of delay are usually examined: fixed time delay and continuously distributed time delay (fixed delay and continuous delay henceforth). The former is applicable in economic situations in which an institutionally or socially determined fixed period of time delay is present for the agents involved. The latter is appropriate for economic situations in which different lengths of delays are distributed over the different agents. Uncertain delays can be modeled by continuous delays, and the same types of models describe the situation when the firms want to react to average past information instead of following sudden market change. So the choice of the type of delays has situation-dependency and results in the use of different analytical tools. In the cases of fixed delays the dynamic equations are delay differential equations where the characteristic equation is a mixed polynomial-exponential equation with infinitely many eigenvalues.[1] The classical book by Bellman and Cooke (1956) offers the theory of such dynamic models. Kuang (1993) gives good theoreical foundation and comprehensive summary of application in population dynamics. In economic dynamics, Howroyd and Russel (1984) construct two linear continuous time dynamic oligopoly models and examine the effect of the delay on stability. Fixed delay dynamics has been investigated in various economic frameworks ranging from micoreconomics (i.e, oligopoly dynamics) to macroeconomics (i.e., business cycle). In the cases of continuous delays the dynamic equations are Volterra type integro-differential equations. Cushing (1977) discusses the mathematical methodology dealing with such dynamics. Invernizzi and Medio (1991) have introduced continuous delays into mathematical economics, and this methodology is later used to examine dynamic oligopolies by Chiarella and Khomin (1996) and Chiarella and Szidarovszky (2001, 2004). Recently Matsumoto (2009) re-examined the classical Goodwin's accelerator business cycle by replacing fixed delay in the original model with continuous delay. Dynamics generated by fixed delay and continuous delay are compared in Matsumoto and Szidarovszky (2010) in which the Goodwin (2D) model, the Kaldor-Kalecki (3D) model and the Cournot oligopoly (4D) model are examined.

In our earlier paper, Matsumoto and Szidarovszky (2011), boundedly rational monopoly is examined with one and two fixed delays. A complete stability analysis is given and it is demonstrated that in the case of locally unstable

[1] A dynamic equation with fixed delays can be called a mixed difference-differential equation. However it is pointed out by Gandolfo (2009) that such terminology is somewhat dated.

monopoly equilibrium only simple dynamics (i.e., limit cycle) can be born when one fixed delay is involved while complex dynamics are reached through a period-doubling bifurcation when two fixed delays are involved. In this paper the fixed delay is replaced with continuous delay and in addition to complete stability analysis, the asymptotic behavior of the equilibrium with fixed and continuous delays will be compared.

This paper is organized as follows. In the next section, we construct a gradient dynamic model of boundedly rational monopoly. Then in Section 3, we analytically examine local dynamics and numerically show that the continuous delay has a threshold value at which the monopoly equilibrium loses stability. In Section 4 we introduce a cautious expectation formation and demonstrate that stability switch occurs twice, one switch to instability from stability for a small delay and the other switch to stability from instability for a large delay. Concluding remarks are given in Section 5.

2 Delay Monopoly

In this section we construct a basic dynamic model of a boundedly rational monopoly which produces output q with marginal cost c. The price function is linear

$$f(q) = a - bq,\ a,\ b > 0.$$

However, the monopoly does not know the true price function due to incomplete information. There are several ways to deal with the behavior under such circumstances. If the monopoly believes in a misspecified price function and chooses its decision accordingly, then a self-confirming steady state may emerge which is different from the stationary state with full information. Or if it does not know certain parameters of the price function, although knowing that it is linear, then the monopoly uses a local linear approximation of the price function based on its past output data to update its estimate.[2] In this study, assuming that the monopoly knows only a finite points of the true price function, we confine attention to a situation in which the monopoly is able to estimate the rate of a profit change by using actual prices it received in the past. The estimated rate at a value q^e of output which is believed to be close to the actual output it would select is given by

$$\frac{d\pi^e}{dq^e} = a - c - 2bq^e.$$

So the approximating gradient dynamics is

$$\dot{q} = \alpha(q)\frac{d\pi^e}{dq^e} \tag{1}$$

where $\alpha(q)$ is an adjustment function and the dot over a variable means a time derivative. In constructing best response dynamics, global information

[2] See Chapter 5 of Bischi *et al.* (2010) for stability/instability of economic models with misspecified and uncertain price functions.

is required about the profit function, however, in applying gradient dynamics, only local information is needed. We make the familiar assumption that the adjustment function has linear dependency on output:

Assumption 1. $\alpha(q) = \alpha q$ with $\alpha > 0$.

The gradient dynamics under Assumption 1 with an expected demand q^{ε} is presented by

$$\dot{q}(t) = \alpha q(t) \left[a - c - 2bq^{\varepsilon}(t)\right] \tag{2}$$

where t denotes a point of continuous time. Since $q(t) = q^{\varepsilon}(t)$ for all t holds at a stationary point, equation (2) has two stationary points; a trivial point $q(t) = 0$ and a nontrivial point

$$q^M = \frac{a-c}{2b}$$

where $a > c$ is assumed to ensure that the nontrivial point is positive. We call q^M a *monopoly equilibrium*. Dynamic behavior of (2) depends on the formation of expectations. In a dynamic model with continuous time scales, time delays can be modeled with fixed delays or continuously delays. Matsumoto and Szidarovszky (2011) examined dynamic monopoly with single and two fixed delays. In this study we adopt a single continuous delay and consider the delay effects on the dynamics.[3] Before proceeding, we briefly summarize the results obtained in the dynamic monopoly with one fixed delay.

We assume $q^e(t) = q(t-\tau)$ where $\tau > 0$ denotes a fixed delay. Substituting $q^e(t)$ for $q(t-\tau)$ gives a nonlinear delay differential equation,

$$\dot{q}(t) = \alpha q(t) \left[a - c - 2bq(t-\tau)\right]. \tag{3}$$

Linearizing equation (3) and introducing the new variable, $x(t) = q(t) - q^M$ yield the following linearized form,

$$\dot{x}(t) = -\gamma x(t-\tau) \text{ with } \gamma = \alpha(a-c) > 0.$$

Substituting the exponential solutions $x(t) = x_0 e^{\lambda t}$ into the linearized equation gives the characteristic equation

$$\lambda + \gamma e^{-\lambda\tau} = 0. \tag{4}$$

The sufficient condition for local asymptotic stability is that the real parts of the eigenvalues are negative. It is shown in Matsumoto and Szidarovszky (2011) that the monopoly equilibrium is locally asymptotically stable for $0 < \tau < \tau^*$, locally unstable for $\tau > \tau^*$ and undergoes a Hopf bifurcation at $\tau = \tau^*$ where a threshold value τ^* of delay is defined by

$$\tau^* = \frac{\pi}{2\alpha(a-c)}.$$

This partition curve, which divides the parameter space into a stable and unstable regions, is downward sloping with respect to $\gamma = \alpha(a-c)$. The monopoly equilibrium is locally stable below the partition curve, locally asymptotically unstable above and bifurcates to a limit cycle when it crosses the curve.

[3] Monopoly dynamics with two continuous delays is considered in the subsequent paper.

3 Single Time Delay I

As it was mentioned earlier, continuous delay is an alternative approach to deal with delays. The gradient dynamics with a single continuous delay is given by the following two equations:

$$\begin{aligned} \dot{q}(t) &= \alpha q(t)\left[a - c - 2bq^{\varepsilon}(t)\right] \\ q^{\varepsilon}(t) &= \int_0^t W(t-s,\tau,m)q(s)ds \end{aligned} \tag{5}$$

where α is the speed of adjustment and the weighting function is defined by

$$W(t-s,\tau,m) = \begin{cases} \dfrac{1}{\tau}e^{-\frac{t-s}{\tau}} & \text{if } m = 0, \\ \dfrac{1}{m!}\left(\dfrac{m}{\tau}\right)^{m+1}(t-s)^m e^{-\frac{m(t-s)}{\tau}} & \text{if } m \geq 1 \end{cases} \tag{6}$$

where m is a nonnegative integer and τ is a positive real parameter, which is associated with the length of the delay. The first equation of (5) implies that the growth rate of output is proportional to the expected demand. The second equation indicates that the expected output at time t is the weighted average of the actual demand in the past. According to (6), the shape of the weighting function is determined by the value of the shape parameter, m. For $m = 0$, weights are exponentially declining with the most weight given to the most current data. For $m \geq 1$, zero weight is given to the most current data, rising to maximum at $s = t - \tau$ and declining exponentially thereafter. The weights take a bell-shaped form which becomes taller and thinner as m increases.

To consider local dynamics of this system in a neighborhood of the equilibrium point, we need to construct a linearized version. If the deviations of the actual and expected outputs from the equilibrium value are denoted by $x_\delta(t) = q(t) - q^M$ and $x_\delta^{\varepsilon}(t) = q^{\varepsilon}(t) - q^M$, then the linearized system with continuous delay can be written as

$$\begin{aligned} \dot{x}_\delta(t) &= -\gamma x_\delta^{\varepsilon}(t), \\ x_\delta^{\varepsilon}(t) &= \int_0^t W(t-s,\tau,m)x_\delta(s)ds, \end{aligned} \tag{7}$$

where $\gamma = 2\alpha bq^M$. To examine dynamics of system (7), we substitute the second equation of (7) into the first to obtain the following Volterra-type integro-differential equation:

$$\dot{x}_\delta(t) + \gamma\int_0^t W(t-s,\tau,m)x_\delta(s)ds = 0.$$

Looking for the solution in the usual exponential form $x_\delta(t) = x_0 e^{\lambda t}$ and substituting it into the above equation, we obtain

$$\lambda + \gamma \int_0^t W(t-s, \tau, m) e^{-\lambda(t-s)} ds = 0.$$

Introducing the new variable $z = t - s$ simplifies the integral as

$$\int_0^t W(t-s, \tau, m) e^{-\lambda(t-s)} ds = \int_0^t W(z, \tau, m) e^{-\lambda z} dz.$$

By letting $t \to \infty$ and assuming that $\mathrm{Re}(\lambda) + \frac{m}{\tau} > 0$, we have

$$\int_0^\infty \frac{1}{\tau} e^{-\frac{z}{\tau}} e^{-\lambda z} dz = (1 + \lambda\tau)^{-1} \text{ if } m = 0$$

and

$$\int_0^\infty \frac{1}{m!} \left(\frac{m}{\tau}\right)^{m+1} z^m e^{-\frac{mz}{\tau}} e^{-\lambda z} dz = \left(1 + \frac{\lambda\tau}{m}\right)^{-(m+1)} \quad \text{if } m \geq 1.$$

That is,

$$\int_0^\infty W(z, \tau, m) e^{-\lambda z} dz = \left(1 + \frac{\lambda\tau}{\bar{m}}\right)^{-(m+1)}$$

with

$$\bar{m} = \begin{cases} 1 & \text{if } m = 0, \\ m & \text{if } m \geq 1. \end{cases}$$

Then the characteristic equation becomes

$$\lambda \left(1 + \frac{\tau}{\bar{m}}\lambda\right)^{m+1} + \gamma = 0. \tag{8}$$

Expanding the characteristic equation presents the $(m+2)$-th order polynomial equation

$$a_0 \lambda^{m+2} + a_1 \lambda^{m+1} + ... + a_{m+1}\lambda + a_{m+2} = 0$$

where the coefficients a_i are given as

$$a_k = \left(\frac{\tau}{\bar{m}}\right)^{m+1-k} \begin{pmatrix} m+1 \\ k \end{pmatrix} \text{ for } 0 \leq k \leq m,$$

$$a_{m+1} = 1 \text{ and } a_{m+2} = \gamma.$$

In the case of the high order polynomial equation, the Routh-Hurwitz theorem[4] provides the necessary and sufficient conditions for all the roots to have negative

[4] See, for example, Gandolfo (2009) for this theorem.

real parts. Applying the theorem, we first need to construct the Routh-Hurwitz determinant:

$$D_{m+2} = \det \begin{pmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ a_7 & a_6 & a_5 & a_4 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{m+2} \end{pmatrix}.$$

Then the stability conditions are as follows:

(1) all coefficients are positive, $a_k > 0$ for $k = 0, 1, 2, ..., m+2$,

(2) the principal minors of the Routh-Hurwitz determinant are all positive,

$$D_{m+2}^2 > 0,\ D_{m+2}^3 > 0, ...,\ D_{m+2}^{m+1} > 0$$

where D_{m+2}^k is the k-th order leading principal minor of D_{m+2}. Notice that $D_{m+2}^{m+1} > 0$ always leads to $D_{m+2}^{m+2} > 0$ since $a_{m+2} = \gamma > 0$.

Since it is difficult to obtain a general solution of equation (8), we draw attention to special cases with $m = 0, 1, 2, 3$ and $m \to \infty$ and examine stability of the monopoly equilibrium analytically as well as numerically.

Case I-0. $m = 0$.

Substituting $m = 0$ reveals that the characteristic equation (8) is quadratic, $\tau\lambda^2 + \lambda + \gamma = 0$ where all coefficients are positive. It does not have nonnegative roots and the real parts of the complex roots are negative. Thus the monopoly equilibrium is locally asymptotically stable for all $\tau > 0$. Since the delay does not affect asymptotic behavior of the monopoly equilibrium, such a delay is called *harmless*.

Case I-1. $m = 1$.

The characteristic equation (8) with $m = 1$ becomes cubic and its coefficients are all positive

$$a_0 = \tau^2 > 0, a_1 = 2\tau > 0, a_2 = 1 > 0, a_3 = \gamma > 0.$$

According to the Routh-Hurwitz criterion, the following leading minor of D_3 needs to be positive for preserving stability of the equilibrium,

$$D_3^2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}.$$

To obtain $D_3^2 = \tau(2 - \tau\gamma) > 0$, the delay τ should be less than the threshold value

$$\tau_1^* = \frac{2}{\gamma}. \tag{9}$$

There is a possibility of the emergence of a limit cycle when loss of stability occurs at $\tau = \tau_1^*$. The Hopf bifurcation theorem comes in to provide the sufficient conditions for it:

(H1) the characteristic equation of the dynamic system has a pair of pure imaginary roots and has no other roots with zero real parts;

(H2) the real parts of these roots vary with a bifurcation parameter.

The $D_3^2 = 0$ curve divides the parameter space into stable and unstable parts. Substituting $a_3 = a_1 a_2 / a_0$ into the characteristic equation gives the factored form

$$(a_1 + a_0\lambda)(a_2 + a_0\lambda^2) = 0.$$

We have therefore three characteristic roots, two purely imaginary roots and one real and negative root,

$$\lambda_{1,2} = \pm\sqrt{-\frac{a_2}{a_0}} = \pm i\frac{1}{\tau} \text{ and } \lambda_3 = -\frac{a_1}{a_0} = -\frac{2}{\tau} < 0.$$

The first condition (H1) of the Hopf theorem is satisfied.

Next we select the delay τ as the bifurcation parameter and consider the roots of the characteristic equation as continuous functions of τ:

$$\tau^2\lambda(\tau)^3 + 2\tau\lambda(\tau)^2 + \lambda(\tau) + \gamma = 0$$

Differentiating it with respect to τ gives

$$\frac{d\lambda}{d\tau} = -\frac{2\tau\lambda^3 + 2\lambda^2}{3\tau^2\lambda^2 + 4\tau\lambda + 1}.$$

Substituting $\lambda = i/\tau$, rationalizing the right hand side and noticing that the terms with λ and λ^3 are imaginary while the constant and the term λ^2 are real yield the following form of the real part of the derivative of λ with respect to τ:

$$\text{Re}\left[\left.\frac{d\lambda}{d\tau}\right|_{\lambda=\frac{i}{\tau}}\right] = \frac{1}{5\tau^2} > 0.$$

The last inequality indicates that the second condition (H2) is also satisfied. The real parts of the complex roots change to positive from negative value resulting in the loss of stability on the partition curve. Hence the Hopf bifurcation theorem confirms the birth of a limit cycle when stability is lost.

We numerically examine the analytical result just obtained. The dynamic system under the investigation is obtained by substituting $m = 1$ into (5),

$$\dot{q}(t) = \alpha q(t)\left[a - c - 2bq^{\varepsilon}(t)\right]$$

$$q^{\varepsilon}(t) = \int_0^t \left(\frac{1}{\tau}\right)^2 (t-s)e^{-\frac{t-s}{\tau}} q(s)ds.$$

Differentiating the second equation with respect to t and introducing a new variable

$$q_0(t) = \int_0^t \frac{1}{\tau} e^{-\frac{t-s}{\tau}} q(s) ds$$

transforms the dynamic system with continuous delay into a 3D system of ordinary differential equations

$$\dot{q}(t) = \alpha q(t) \left[a - c - 2bq^{\varepsilon}(t) \right],$$

$$\dot{q}^{\varepsilon}(t) = \frac{1}{\tau} \left(q_0(t) - q^{\varepsilon}(t) \right),$$

$$\dot{q}_0(t) = \frac{1}{\tau} \left(q(t) - q_0(t) \right).$$

We specify the parameters' values as $a = 2,\ b = 1,\ c = 1,\ \alpha = 1,$[5] and take $\tau = 3$ and the initial values of all variables to be $q^M - 0.1$. Then simulating the 3D system exhibits the birth of a limit cycle as illustrated in Figure 1 where a black trajectory starting at the black dot (i.e., positive initial point) converges to a red cycle in the $(q,\ q^{\varepsilon},\ q_0)$ space. We then summarize this result as follows: the monopoly equilibrium point with $m = 1$ is destabilized through a Hopf bifurcation and, as it is numerically confirmed, converges to a limit cycle when the delay τ is larger than the critical value $2/\gamma$.

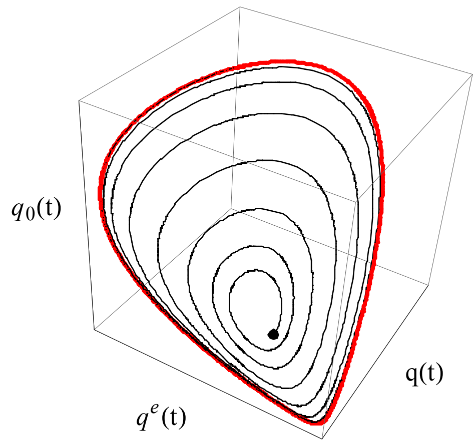

Figure 1. Convergence to a limit cycle in the $(q, q^{\varepsilon}, q_0)$ plane

Case I-2. $m = 2$.

The characteristic equation (8) with $m = 2$ becomes quartic

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

[5] This set of the parameters are repeatedly used in the following numerical examples. Notice that $\gamma = 1$ under this set.

and its coefficients are all positive,

$$a_0 = \tau^3 > 0, a_1 = 6\tau^2 > 0, a_2 = 12\tau > 0, a_3 = 8 > 0, a_4 = 8\gamma > 0.$$

The principal minors of the Routh-Hurwitz determinant are $D_4^2 = 64\tau^3 > 0$ and $D_4^3 = 32\tau^3(16 - 9\gamma\tau)$. To obtain $D_3^2 > 0$, the delay τ should be less than the threshold value

$$\tau_2^* = \frac{16}{9\gamma} \simeq \frac{1.78}{\gamma}. \tag{10}$$

Thus the equilibrium is locally asymptotically stable if $\tau < \tau_2^*$ and locally unstable if $\tau > \tau_2^*$. In the same way as in Case I-1, we can show the existence of a limit cycle at the critical value τ_2^*. In particular, solving $D_4^3 = a_1a_2a_3 - (a_0a_3^2 + a_1^2a_4) = 0$ for a_4, substituting it into the characteristic equation and factoring the resultant equation yield

$$(a_3 + a_1\lambda^2)(a_1a_2 - a_0a_3 + a_1^2\lambda + a_0a_1\lambda^2) = 0.$$

The solutions of $a_3 + a_1\lambda^2 = 0$ are purely imaginary,

$$\lambda_{1,2} = \pm i\frac{2}{\tau\sqrt{3}},$$

and the other two characteristic roots are the solutions of the quadratic equation $(a_1a_2 - a_0a_3) + a_1^2\lambda + a_0a_1\lambda^2 = 0$,

$$\lambda_{3,4} = \frac{-9 \pm i\sqrt{15}}{3\tau}$$

whose real parts are negative. The first condition (H1) of the Hopf bifurcation theorem is satisfied.

To confirm the second condition, we choose τ as the bifurcation parameter again and differentiate the characteristic equation with respect to τ to have

$$\frac{d\lambda}{d\tau} = -\frac{3\tau^2\lambda^4 + 12\tau\lambda^3 + 12\lambda^2}{4\tau^3\lambda^3 + 18\tau^2\lambda^2 + 24\tau\lambda + 8}.$$

Substituting $\lambda = i\frac{2}{\tau\sqrt{3}}$ and taking the real part, we have

$$\mathrm{Re}\left[\left.\frac{d\lambda}{d\tau}\right|_{\lambda = i\frac{2}{\tau\sqrt{3}}}\right] = \frac{6}{19\tau^2} \simeq \frac{0.316}{\tau^2} > 0.$$

We thereby confirm that the monopoly equilibrium with $m = 2$ is destabilized through a Hopf bifurcation when the delay τ crosses the critical value τ_2^*.

Case I-3. $m = 3$.

The characteristic equation (8) is qintic and its coefficients are all positive

$$a_0 = \tau^4 > 0, a_1 = 12\tau^3 > 0, a_2 = 54\tau^2 > 0, a_3 = 108\tau > 0, a_4 = 81 > 0,\ a_5 = 81\gamma > 0.$$

The first two principal minors of the Routh-Hurwitz determinant are positive

$$D_5^2 = 540\tau^5 > 0 \text{ and } D_5^3 = 972\tau^6(48 + \gamma\tau) > 0.$$

The sign of the fourth order principal minor $D_5^4 = -6561\tau^6(\gamma^2\tau^2 + 336\gamma\tau - 576)$ is ambiguous. To obtain $D_5^4 > 0$, the delay τ should be less than the threshold value

$$\tau_3^* = \frac{24(5\sqrt{2} - 7)}{\gamma} \simeq \frac{1.71}{\gamma}. \tag{11}$$

Although we omit the detail, we can also show that the continuous delay system (5) with $m = 3$ can generate a limit cycle through a Hopf bifurcation in the same way as in the previous cases when the monopoly equilibrium loses stability.

The relations (9), (10) and (11) define the partition curves of (γ, τ) that divide the (γ, τ) space into stable and unstable parts. The three partition curves for $m = 1, 2, 3$ and the stability (yellow) region with fixed delay defined by $\tau\gamma < 2$ are depicted in Figure 2. It can be seen that all curves are hyperbolic and are approaching the red-colored boundary of the stability region from above. In other words, the stable region with continuous delay becomes smaller as the value of m increases and converges to the region defined by the fixed delay when m tends to infinity. This result obtained are natural if we notice the properties of the weighting function. The weighting function for $m \geq 1$ is a bell-shaped and becomes more peaked around $t - s$ as m increases. Furthermore it tends to the Dirac delta function if $m \to \infty$. In consequence, for sufficiently large m, the weighting function may be regarded as very close to the Dirac delta function and the dynamic behavior under the continuous delay is very similar to that under the fixed delay. We can explain this phenomenon mathematically by noticing that the characteristic equation (8) of the continuously distributed case can be written as

$$\lambda + \gamma\left(1 + \frac{\tau\lambda}{m}\right)^{-(m+1)}\left(1 + \frac{\tau\lambda}{m}\right)^{-1} = 0,$$

and as $m \to \infty$, the left hand side converges to

$$\lambda + \gamma e^{-\lambda\tau} = 0.$$

This is the characteristic equation of the delay differential equation with a single fixed delay and is identical with equation (4). In short, under continuous delay, although we comprehensively use all delayed or past output data, the stability domain is sensitive to the shape of the weighting function. Hence we obtain the following two results:

Proposition 1 *(1) The monopoly equilibrium of the continuously distributed single delay model is always stable for any delays if* $m = 0$ *and is destabilized through a Hopf bifurcation if* $m \geq 1$; *(2) The stability region decreases as* m *increases (i.e.,* $\tau^* < \tau_3^* < \tau_2^* < \tau_1^*$*) and converges to the stability region obtained under the fixed delay when* m *goes to infinity (i.e.,* $\tau_m^* \to \tau^*$ *as* $m \to \infty$*).*

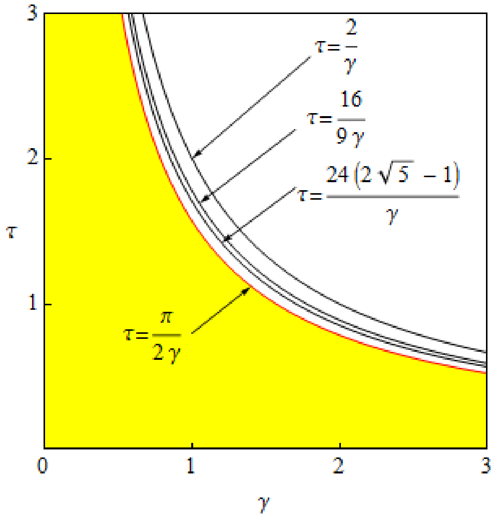

Figure 2. Three partiton curves and stability region

4 Single Time Delay II

In this section, we introduce a *cautious* expectation formation defined by

$$q^{\varepsilon}(t) = \omega q^{e}(t) + (1-\omega)q(t)$$

$$q^{e}(t) = \int_0^t W(t-s,\tau,m)q(s)ds \tag{12}$$

with $0 < \omega \leq 1$. The expectation is formed with two steps: the weighted average of the past data is calculated at the first step and then the expected demand is selected, at the second step, somewhere in between the current output level and the weighted average level.[6] If $m \geq 1$, then zero weight is given to the most current data in the weighting function, and so the second term in the first equation of (12) gives a larger weight to it taking a certain learning procedure based on past data at the second stage. We examine in some detail the dynamic effects caused by a single delay with the cautious expectation formation. Following the method we take in the previous section, the characteristic equation of the system (12) can be obtained as

$$\lambda\left(1+\frac{\lambda\tau}{\bar{m}}\right)^{m+1} + \gamma\left[\omega + (1-\omega)\left(1+\frac{\lambda\tau}{\bar{m}}\right)^{m+1}\right] = 0 \tag{13}$$

[6] Since the first equation of (12) can be rewritten as

$$q^{\varepsilon}(t) - q(t) = \omega(q^{e}(t) - -q(t)).$$

It can be mentioned that the expected demand is formed in such a way that the expectation error is proportional to the difference between the weighted average level and the current level.

which is reduced to equation (8) if $\omega = 1$. To find how the shape of the weighting function, $W(t-s, \tau, m)$, affects the dynamics of q^M, we sequentially increase the value of m from zero to five and then to infinity.

Case II-0 $m = 0$

Substituting $m = 0$ in equation (13) presents the form

$$\lambda(1+\lambda\tau) + \gamma\omega + \gamma(1-\omega)(1+\lambda\tau) = 0$$

or

$$\tau\lambda^2 + (1 + \tau\gamma(1-\omega))\lambda + \gamma = 0.$$

Since all coefficients are positive, there is no nonnegative root and the real parts of the complex eigenvalues are negative,

$$\mathrm{Re}(\lambda_\pm) = -\frac{1+(1-\omega)\tau\gamma}{2\tau} < 0,$$

implying that the equilibrium is locally asymptotically stable for all $\tau > 0$. As in Case I-0, the coninuous delay is again harmless when the weight exponentially declines (i.e., $m = 0$).

For $m \geq 1$, expanding the characteristic equation (13) yields the polynomial equation of degree $m + 2$

$$b_0\lambda^{m+2} + b_1\lambda^{m+1} + \cdots + b_m\lambda^2 + b_{m+1}\lambda + b_{m+2} = 0 \tag{14}$$

where the coefficients are defined by

$$b_0 = a_0,$$

$$b_k = a_k + \gamma(1-\omega)a_{k-1} \text{ for } 1 \leq k \leq m,$$

$$b_{m+1} = a_{m+1} + \gamma(1-\omega)a_m,$$

$$b_{m+2} = \gamma$$

with

$$a_k = \left(\frac{\tau}{m}\right)^{m+1-k} \begin{pmatrix} m+1 \\ k \end{pmatrix} \text{ for } 0 \leq k \leq m+1. \tag{15}$$

Case II-1 $m = 1$

Equation (13) with $m = 1$ becomes cubic:

$$b_0\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{16}$$

where the coefficients are defined by

$$b_0 = \tau^2,\ b_1 = 2\tau + \gamma(1-\omega)\tau^2,\ b_2 = 1 + 2\tau\gamma(1-\omega) \text{ and } b_3 = \gamma.$$

All coefficients are positive, so the Routh-Hurwitz criterion implies that the monopoly equilibrium is locally asymptotically stable if

$$D_3^2 = \det\begin{pmatrix} b_1 & b_0 \\ b_3 & b_2 \end{pmatrix} > 0 \tag{17}$$

where the determinant is written as

$$\tau\left(2\gamma^2(1-\omega)^2\tau^2 + (4\gamma - 5\gamma\omega)\tau + 2\right).$$

Since $\tau > 0$, this expression is positive if and only if

$$f(\tau\gamma) = 2(1-\omega)^2(\tau\gamma)^2 + (4-5\omega)\tau\gamma + 2 > 0. \tag{18}$$

$f(\tau\gamma)$ is quadratic with respect to $\tau\gamma$ and its discriminant has the form

$$D = \omega(9\omega - 8).$$

If $\omega < 8/9$, then $D < 0$ so (18) always holds. If $\omega = 8/9$, then the right hand side of (18) simplifies as

$$f(\tau\gamma) = 2\left(\frac{\tau\gamma}{9} - 1\right)^2$$

so (18) holds if $\tau\gamma \neq 9$. If $\omega > 8/9$, then $D > 0$ and $f(\tau\gamma)$ has two distinct real roots:

$$\tau\gamma_A = \frac{5\omega - 4 - \sqrt{\omega(9\omega-8)}}{4(1-\omega)^2} \text{ and } \tau\gamma_B = \frac{5\omega - 4 + \sqrt{\omega(9\omega-8)}}{4(1-\omega)^2}.$$

$f(0) > 0$ and $f'(0) < 0$ imply that both roots are positive. So the monopoly equilibrium is locally asymptotically stable if $\omega > 8/9$ and $\tau\gamma < \tau\gamma_A$ or $\tau\gamma > \tau\gamma_B$.

These analytical results are graphically visualized in Figure 3. The locus of $b_1b_2 - b_3b_0 = 0$ is depicted as the blue-red curve passing through points A, C and B and partitions the nonnegative $(\omega,\ \tau\gamma)$ plane into a stable (white) region and a unstable (orange) region. The monopoly equilibrium is locally asymptotically stable regardless of the value of $\tau\gamma$ if $\omega < 8/9$. Stability-switch occurs twice if $8/9 < \omega < 1$. The vertical real line at $\omega = \bar{\omega} > 8/9$ crosses the partition curve at two points A and B whose ordinates are $\tau\gamma_A$ and $\tau\gamma_B$, respectively. The monopoly equilibrium is locally stable for $\tau = 0$, loses stability at point A and gains stability at point B when $\tau\gamma$ increases from zero along the vertical line at $\omega = \bar{\omega}$. For $\omega = 1$, the stability condition (18) is reduced to

$$-\tau\gamma + 2 > 0.$$

The monopoly equilibrium is locally stable if $\tau\gamma < 2$ and locally unstable if $\tau\gamma > 2$, implying that the stability switch occurs only once at $\tau\gamma = 2$ for $\omega = 1$, which is the same as the result obtained in Case I-1. This coincidence is reasonable since Case I-1 is identical with Case II-1 for $\omega = 1$.

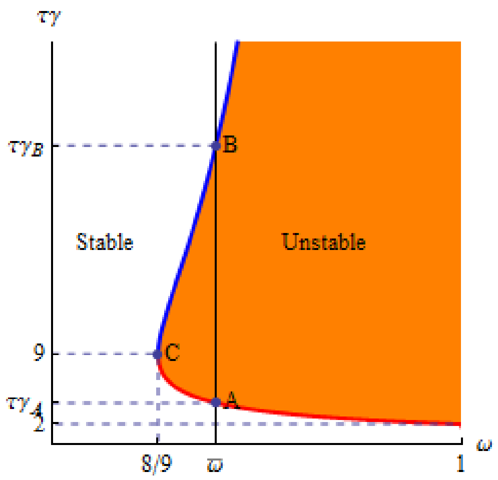

Figure 3. Stability region with $m = 1$

The local behavior of the equilibrium at points A and B has been analytically examined. However, global dynamic behavior of the locally unstable equilibrium between points A and B is still in question. To investigate such behavior, we will show that at these critical values Hopf bifurcation occurs giving the possibility of the birth of limit cycles even under the cautious expectation formation.

We start with the first condition, (H1). The cubic characteristic equation (16) can be factored when $b_1 b_2 - b_0 b_3 = 0$,

$$(b_1 + b_0\lambda)(b_2 + b_0\lambda^2) = 0.$$

This factorization implies that there are a pair of purely imaginary roots and one negative root. The two purely imaginary roots are given by

$$\lambda_{1,2} = \pm\sqrt{-\frac{b_2}{b_0}} = \pm i\beta$$

with

$$\beta = \frac{\sqrt{1 + 2\tau\gamma(1-w)}}{\tau}$$

and the negative root by

$$\lambda_3 = -\frac{b_1}{b_0} = -\frac{2 + (1-w)\tau\gamma}{\tau}.$$

The fulfilment of (H1) is confirmed.

We turn to verification of (H2). Selecting τ as the bifurcation parameter we might treat the eigenvalue as a continuous function of τ, $\lambda = \lambda(\tau)$. Differentiating the characteristic equation (16) implicitly with respect to τ and arranging terms, we have

$$\frac{d\lambda}{d\tau} = -\frac{2\lambda^3\tau + \lambda^2(2 + 2\tau\gamma(1-\omega)) + 2\lambda\gamma(1-\omega)}{3\lambda^2\tau^2 + 2\lambda(2\tau + \gamma(1-\omega)\tau^2) + (1 + 2\tau\gamma(1-\omega))}. \tag{19}$$

At $\lambda = i\beta$,

$$\frac{d\lambda}{d\tau} = \frac{(2\tau\beta^3 - 2\beta\gamma(1-\omega))i + \beta^2(2 + 2\tau\gamma(1-\omega))}{-2\beta^2\tau^2 + 2\beta i(2\tau + \gamma(1-\omega)\tau^2)}$$

where the relation $(\beta\tau)^2 = 1 + 2\tau\gamma(1-\omega)$ is used to simplify the denominator of the above equation. Then the real part becomes

$$\mathrm{Re}\left[\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\beta}\right] = \frac{1 - \gamma^2\tau^2(1-\omega)^2}{\beta^2\tau^4 + (2\tau + \gamma(1-\omega)\tau^2)^2}$$

with the positive denominator. It is easy to see that

$$\tau\gamma_A < \frac{1}{1-\omega} < \tau\gamma_B,$$

so

$$\mathrm{Re}\left[\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\beta}\right] > 0 \text{ at point } A \text{ and } \mathrm{Re}\left[\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\beta}\right] < 0 \text{ at point } B.$$

So at point A, the real part changes sign from negative to positive and at point B, from positive to negative. This demonstrates that (H2) of the Hopf theorem is satisfied. So Hopf bifurcation occurs at both points.

We next numerically examine the switching of stability and global behavior. The dynamic system under the investigation (i.e. $m = 1$) is obtained as,

$$\begin{aligned} \dot{q}(t) &= \alpha q(t)\left[a - c - 2b\left(\omega q^{\varepsilon}(t) + (1-\omega)q(t)\right)\right] \\ q^{\varepsilon}(t) &= \int_0^t \frac{1}{\tau^2}(t-s)e^{-\frac{t-s}{\tau}}q(s)ds. \end{aligned} \tag{20}$$

Differentiating the second equation with respect to t and introducing a new variable

$$q_0(t) = \int_0^t \frac{1}{\tau}e^{-\frac{t-s}{\tau}}q(s)ds$$

transform the dynamic system with continuously distributed time delay into a 3D system of the ordinary differential equations

$$\begin{aligned}
\dot{q}(t) &= \alpha q(t)\left[a - c - 2b\left(\omega q^{\varepsilon}(t) + (1-\omega)q(t)\right)\right], \\
\dot{q}^{\varepsilon}(t) &= \frac{1}{\tau}\left(q_0(t) - q^{\varepsilon}(t)\right), \\
\dot{q}_0(t) &= \frac{1}{\tau}\left(q(t) - q_0(t)\right).
\end{aligned} \tag{21}$$

We use the same parameter set (i.e., $a = 2,\ b = 1,\ c = 1,\ \alpha = 1$) and the same initial values (i.e., $q(0) = q^{\varepsilon}(0) = q_0(t) = q^M - 0.1$) as in Case I-1. Increasing the value of $\tau\gamma$ from $\tau\gamma_A$ to $\tau\gamma_B$ along the vertical line $\bar{\omega} = 0.91$ in Figure 3, we obtain the bifurcation diagram illustrated in Figure 4 where the local maximum and minimum are plotted against each value of $\tau\gamma$. It can be seen that the monopoly equilibrium loses stability bifurcating to a limit cycle when $\tau\gamma$ increases to $\tau\gamma_A$ and gains stability when $\tau\gamma$ arrives at $\tau\gamma_B$. It is also observed that the amplitude of the limit cycle first increases and then decreases as $\tau\gamma$ increases from $\tau\gamma_A$ to $\tau\gamma_B$.

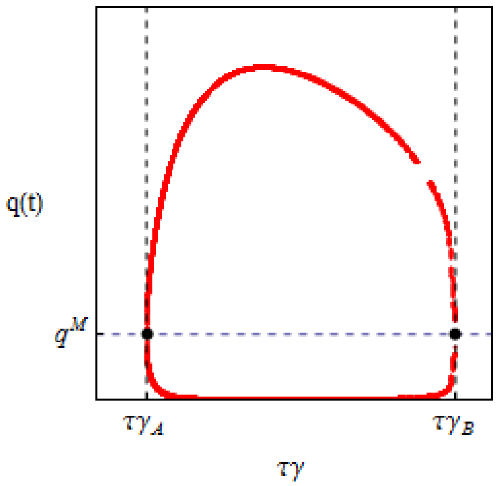

Figure 4. Bifurcation diagram along the $\omega = 0.91$ line

We summarize the results obtained in Case II-1 as follows:

Proposition 2 *Under the cautious expectation formation, dynamics of the monopoly equilibrium* q^M *with* $\tau > 0,\ m = 1$ *and* $0 < \omega \leq 1$ *takes one of the following alternative behavior;*

(I) q^M *is locally asymptotically stable if* $0 < \omega < 8/9$ *regardless of the values of* $\tau\gamma$ *(i.e., the delay is harmless)*;

(II) *q^M bifurcates to a limit cycle at $\tau\gamma_A(\omega)$ and the limit cycle de-bifurcates to the monopoly equilibrium at $\tau\gamma_B(\omega)$ if $8/9 < \omega < 1$ where $\tau\gamma_A(\omega)$ and $\tau\gamma_B(\omega)$ are the bifurcation values depending on ω (i.e., stability switch occurs twice)*;

(III) *q^M bifurcates to a limit cycle at $\tau\gamma = 2$ and never regains stability for $\tau\gamma > 2$ if $\omega = 1$ (i.e., stability switch occurs once).*

Case II-2 $m \geq 2$

As in the same way as in Case II-1, we can check the stability condition, the birth of a limit cycle and stability switch in the case of $m \geq 2$. For example, the characteristic equation (13) with $m = 2$ is quartic,

$$b_0\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0$$

and the stability conditions are given by

$$D_4^2 = \det\begin{pmatrix} b_1 & b_0 \\ b_3 & b_2 \end{pmatrix} > 0 \text{ and } D_4^3 = \begin{pmatrix} b_1 & b_0 & 0 \\ b_3 & b_2 & b_1 \\ 0 & b_4 & b_3 \end{pmatrix} > 0$$

where the coefficients are given by

$$b_0 = \frac{\tau^3}{8},\ b_1 = \frac{3\tau^2}{4} + \frac{\tau^3\gamma(1-\omega)}{8},$$

$$b_2 = \frac{3\tau}{2} + \frac{3\tau^2\gamma(1-\omega)}{4},\ b_3 = 1 + \frac{3}{2}\tau\gamma(1-\omega),\ b_4 = \gamma.$$

Since all coefficients are positive and $D_4^3 > 0$ implies $D_4^2 > 0$, it remains to check whether D_4^3 can be positive. Notice that

$$D_4^3 = \alpha_0(\tau\gamma)^3 + \alpha_1(\tau\lambda)^2 + \alpha_2(\tau\gamma) + \alpha_3 \tag{22}$$

where

$$\begin{aligned} \alpha_0 &= (9\omega - 8)(1-\omega)^2 \\ \alpha_1 &= -12(4 - 9\omega + 5\omega^2) \\ \alpha_2 &= -12(8 - 11\omega) \\ \alpha_3 &= -64 \end{aligned} \tag{23}$$

The discriminant of the cubic equation (22) has the form

$$\Delta = 442368(1-\omega)^3\omega^2(5\omega - 4)$$

which is obtained by substituting α_i of (23) into the definition of the discriminant, $-4\alpha_0\alpha_2^3+\alpha_1^2\alpha_2^2-4\alpha_1^3\alpha_3+18\alpha_0\alpha_1\alpha_2\alpha_3-27\alpha_0^2\alpha_3^2$. If $\omega < 4/5$, then $\Delta < 0$ implying $D_4^3 > 0$ where (23) has a pair of conjugate complex roots and one negative real root. If $\omega = 4/5$, then $\Delta = 0$ implying that $D_4^3 = (10-\tau\gamma)^2(20+\tau\gamma) > 0$ for $\tau\gamma \neq 10$ where (23) has equal roots at the red point on the $D_4^3 = 0$ locus. If $\omega > 4/5$, then $\Delta > 0$ implying that $D_4^3 = 0$ has three distinct real roots. Since equation (22) is cubic, it is possible to derive explicit forms of the real roots. However, to simplify the analysis, we numerically obtain the roots. Taking $\omega_0 = 0.815$ in addition to the parametric set, we have two real roots, $\tau\gamma_A \simeq 5.42$, $\tau\gamma_B \simeq 23.75$ and one negative root. As seen in Figure 5, $D_4^3 \leq 0$ for $\tau\gamma \in [\tau\gamma_A, \tau\gamma_B]$ which is an unstable interval and $D_4^3 > 0$ for $0 \leq \tau\gamma < \tau\gamma_A$ or $\tau\gamma > \tau\gamma_B$. Stability switch occurs twice at $\tau\gamma = \tau\gamma_A$ and $\tau\gamma = \tau\gamma_B$. Further the cubic equation is reduced to a quadratic equation for $\omega = 8/9$ implying that the locus of $D_4^3 = 0$ is defined only for $\omega < 8/9$ and asymptotic to the vertical line at $\omega = 8/9$. If $8/9 < \omega < 1$, then the cubic equation (22) has one positive root $\tau\gamma_C$ for $\omega = \omega_1$. It is confirmed that $D_4^3 > 0$ for $\tau\gamma < \tau\gamma_C$ and $D_4^3 \leq 0$ otherwise. The equilibrium point switches to be unstable at point C on the $D_4^3 = 0$ locus when the delay is increased along the vertical line at $\omega = \omega_1$. The dark gray region is the unstable region for $m = 1$ and adding the light gray region to it gives the unstable region for $m = 2$.

We turn now to show that Hopf bifurcation can occur on the curve $D_4^3 = 0$. When $D_4^3 = 0$ holds, the characteristic equation is factored as

$$(b_3 + b_1\lambda^2)(b_1b_2 - b_0b_3 + b_1^2\lambda + b_0b_1\lambda^2) = 0.$$

It is clear that the characteristic equation has a pair of purely imaginary roots,

$$\lambda_{1,2} = \pm\sqrt{-\frac{b_3}{b_1}} = \pm i\beta$$

with

$$\beta = \sqrt{\frac{4(2+3\gamma(1-\omega)\tau)}{6\tau^2+\gamma(1-\omega)\tau^3}}$$

and the real parts of other two roots are not zero. So condition (H1) is satisfied.

Assuming that the characteristic root depends on τ and then differentiating the characteristic equation with respect to τ, we obtain the derivative:

$$\frac{d\lambda}{d\tau} = -\frac{\frac{3\tau^2}{8}\lambda^4 + \left(\frac{3\tau}{2} + \frac{3\gamma(1-\omega)\tau^2}{8}\right)\lambda^3 + \left(\frac{3}{2} + \frac{3\gamma(1-\omega)\tau}{2}\right)\lambda^2 + \frac{3\gamma(1-\omega)}{2}\lambda}{\frac{\tau^3}{2}\lambda^3 + 3\left(\frac{3\tau^2}{4} + \frac{\gamma(1-\omega)\tau^3}{8}\right)\lambda^2 + 2\left(\frac{3\tau}{2} + \frac{3\gamma(1-\omega)\tau^2}{4}\right)\lambda + \frac{3\gamma(1-\omega)}{2} + 1}.$$

To determine stability switch, we evaluate the derivative at the purely imaginary solution, $\lambda = i\beta$. Thus

$$\mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\beta}\right) = \frac{3(2-(1-\omega)\tau\gamma)}{4+\tau^2(16\beta^2+9\gamma^2(1-\omega)^2+12\gamma(1-\omega)\tau)}.$$

We numerically confirm that the real part of the derivative is positive at point $\tau\gamma_A$ and negative at point $\tau\gamma_B$. So he second condition (H2) is also satisfied. Therefore Hopf bifurcation occurs at both critical points.

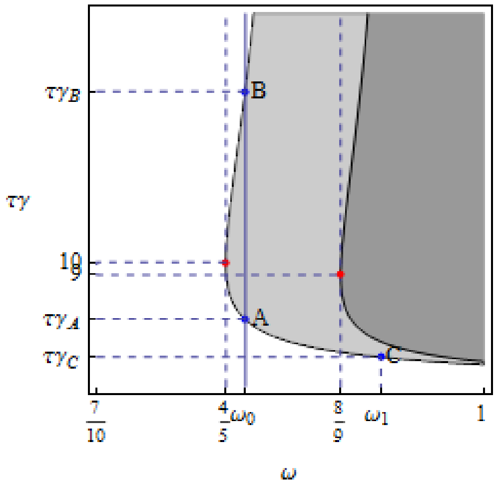

Figure 5. The partition curves with $m = 1$ and $m = 2$

In the same way, it can be numerically confirmed that the stability condition is given by $D_{m+2}^{m+1} > 0$ for $2 \leq m \leq 5$. In Figure 6, five partition curves $D_{m+2}^{m+1} = 0$ for $m = 1, 2, 3, 4, 5$ are depicted. The right most curve is the locus of $D_3^2 = 0$ (i.e., $m = 1$) and the left most curve is the locus of $D_7^6 = 0$ (i.e., $m = 5$). The partition curve shifts leftward with the increasing value of m. At the red dot on each curve, equation $D_{m+2}^{m+1} = 0$ has two real and equal roots.[7] As is shown in Case II-1, for any $m \geq 2$, stability switch occurs twice for ω greater than the abscissa of the red point while the delay becomes harmless for smaller values of ω.

Case II-3 $m \to \infty$

As m increases, the weighting function becomes more peaked around $t - s$ and tends to the Dirac delta function. Notice that as m goes to infinity, the first equation of (20) is reduced to a delay differential equation,

$$\dot{q}(t) = \alpha q(t) \left[a - c - 2b \left(\omega q(t - \tau) + (1 - \omega) q(t)\right)\right] \tag{24}$$

and the characteristic equation (13) converges to

$$\lambda + \gamma(1 - \omega) + \gamma\omega e^{-\lambda\tau} = 0$$

[7] The curves look like steeply-shaped hypabolas. However, if we enlarge each curve in the neighborhood of its red point, then it can be found that the curves take the distorted C-shaped profiles as the curve in Figure 3 or Figure 5.

which is the characteristic equation of the delay differential equation (24). To verify the possibility of stability switch for which the characteristic equation must have a pair of purely imaginary conjugate roots, we can assume without loss of generality that $\lambda = i\upsilon,\ \upsilon > 0$. By the real and imaginary parts, the characteristic equation is divided into two equations

$$\begin{aligned} &\gamma(1-\omega) + \gamma\omega\cos\upsilon\tau = 0, \\ &\upsilon - \gamma\omega\sin\upsilon\tau = 0. \end{aligned} \tag{25}$$

By moving $\gamma(1-\omega)$ and υ to the right hand side of equations in (25), squaring and adding them together, we obtain

$$\upsilon^2 = \gamma^2(2\omega - 1)$$

which is defined only for $\omega > 1/2$, otherwise no stability switch occurs. We then think of the roots as continuous functions in terms of τ and then differentiate the characteristic equation with respect to τ to obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{e^{\tau\lambda}}{\gamma\omega\lambda} - \frac{\tau}{\lambda} \text{ and } e^{\tau\lambda} = -\frac{\gamma\omega}{\lambda + \gamma(1-\omega)}.$$

Thus

$$\begin{aligned} \left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\lambda = i\upsilon} &= \operatorname{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda = i\upsilon}\right)^{-1} \\ &= \frac{1}{\upsilon^2 + \gamma^2(1-\omega)^2} > 0 \end{aligned}$$

The last inequality implies that all the roots that cross the imaginary axis at $i\upsilon$ cross from left to right as τ increases.

From (25), we have

$$\begin{aligned} &\gamma\omega\cos\upsilon\tau = -\gamma(1-\omega), \\ &\gamma\omega\sin\upsilon\tau = \upsilon. \end{aligned} \tag{26}$$

Hence there is a unique $\upsilon\tau$, $\pi/2 < \upsilon\tau < \pi$ such that $\upsilon\tau$ makes both equations in (26) hold. Using the first equation we derive the partition curve

$$\gamma\tau = \frac{\cos^{-1}\left(-\dfrac{1-\omega}{\omega}\right)}{\sqrt{2\omega - 1}}$$

which is defined for $\omega > 1/2$. In Figure 6, in addition to the five partition curves, the downward sloping hyperbolic red partition curve of the fixed delay case are illustrated. The monopoly equilibrium with the fixed delay is locally asymptotically stable in the yellow region. We can summarize three results obtained in Cases II-2 and II-3:

Proposition 3 *In the case with $\tau > 0$ and $m \geq 1$, (1) increasing m has a destabilizing effect in the sense that it decreases the stability region; (2) the stability region with continuously distributed time delay is larger than the one with fixed time delay and the former converges to the latter as m goes to infinity; (3) the stability switch, if possible, occurs twice, implying that the equilibrium is locally stable for smaller or larger values of $\tau\gamma$ while it bifurcates to a limit cycle for medium values.*

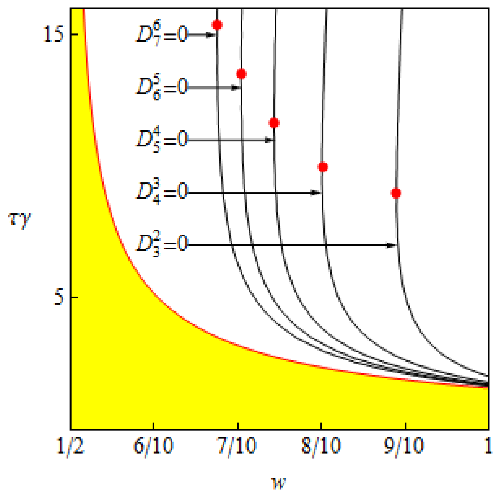

Figure 6. Stability region and five partition cuves

5 Concluding Remarks

In this paper a boundedly rational monopoly with a continuously distributed time delay is examined. Constructing a gradient dynamic system where the rate of the output change is proportional to the derivative of the expected profit, three main results are analytically and numerically demonstrated. First, the stability region depends on the shape parameter of the weighting function of the past data: stability is preserved if the weights exponentially decline while it can be lost if the weighting function takes a bell-shaped profile. Further note that the stability region of the continuously distributed time delay converges to the stability region of the fixed time delay when the shape parameter goes to infinity (in other word, the weighting function converges to the Dirac delta function). Second, the stability switches indicate that the delay has a destabilizing effect. Specifically the stable equilibrium point becomes unstable and never regains stability if the expected demand is formed based only on past data. However, under the cautious expectation formation where the expected demand is a weighted average of the realized output and the past data, switches to instability from stability and then to stability from instability are both possible, depending on the value of the delay. Finally the equilibrium point bifurcates to a limit cycle through Hopf bifurcation when it loses stability.

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