Price and Quantity Competition in Differentiated Oligopoly Revisited*

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Abstract

This study reconsiders the nature of competition in Bertrand and Cournot markets from statics and dynamic points of view. It formalizes optimal behavior in the n-firm framework with product differentiation. Our first findings is that differentiated Bertrand and Cournot equilibria can be destabilized when the number of the firms is greater than three. This finding extends the well-known stability result shown by Theocharis (1960) in which the stability of a non-differentiated Cournot equilibrium is confirmed only in the duopoly framework. A complete analysis is then given in comparing Bertrand and Cournot outputs, prices and profits. The focus is placed upon the effects caused by increasing number of the firms. Our second finding exhibits that the number of the firms really matters in the comparison. In particular, it demonstrates that the comparison results obtained in the duopoly framework do not necessarily hold in the general n-firm framework. This finding extends the results shown by Singh and Vivies (1984) that examine the duality of these two competitions in the duopoly markets and complements the analysis developed by Häcker (2000) that makes comparison in the n-firm markets.

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1 Introduction

So far, a considerable number of studies has been made on the nature of Cournot and Bertrand competitions in which the firms are adjusting their produced quantities and prices, respectively. In differentiated Bertrand and Cournot markets using the duopoly framework, Singh and Vives (1984) show, among others, the followings clear-cut results\(^1\):

(i-SV) prices are higher and quantities lower under Cournot competition than under Bertrand competition regardless of whether the goods are substitutes or complements;

(ii-SV) Cournot competition is more profitable than Bertrand competition if the goods are substitutes;

(iii-SV) Bertrand competition is more profitable than Cournot competition if the goods are complements.

In \(n\)-firm differentiated oligopoly markets, Häckner (2000) points out that some of Singh and Vives’ results are sensitive to the duopoly framework: although (iii-SV) is robust in the \(n\)-firm framework, (i-SV) and (ii-SV) can be reversed in the \(n\)-firm framework with \(n > 2\). In particular, it is shown that prices can be higher under Bertrand competition than under Cournot competition when the goods are complements and Bertrand competition can be more profitable when the goods are substitutes.

In this study, adopting the \(n\)-firm framework, we shall look more carefully into the results developed by Häckner (2000) from dynamic and economic points of view. It has been, since Theocharis (1960), well-known that if the number of firms is more than three, then the Cournot equilibrium becomes unstable even in a linear structure where the demand and the cost functions are linear. This controversial result is shown when the goods are homogenous (i.e., non-differentiated). Considering the two types of competitions with \(n\) firms, we may raise a natural question whether the similar result holds or not when the goods are differentiated, namely, whether the differentiated Cournot and Bertrand equilibria become destabilized by the increasing number of firms. We answer the question in the affirmative. The Cournot equilibrium is possibly destabilized when the goods are substitutes and remains stable when the goods are complements. On the other hand, the Bertrand equilibrium is possibly destabilized when the goods are complements and it is not when the goods are substitutes.

In the linear structure, it is not difficult to convert an inverse demand function into a direct demand function. To make the direct demand economically meaningful, it is usually, but implicitly, assumed that its non-induced or price independent demand is positive. This assumption is explicitly considered in Singh and Vives (1984). However, its role is not examined in Häckner (2000).

\(^1\)SV stands for Singh and Vives. MS to be appeared in the later part of the Introduction means Matsumoto and Szidarovszky.
(i-MS) Bertrand price can be higher than Cournot price when the goods are complements whereas Cournot output can be larger than Bertrand output when the goods are substitutes;

(ii-MS) Bertrand profit can be higher than Cournot profit when the goods are substitutes;

(iii-MS) Cournot profit can be higher than Bertrand profit when the goods are complements.

The rest of the paper is organized as follows. In Section 2 we present an $n$-firm linear oligopoly model and determine the firm’s optimal behavior under Cournot and Bertrand competitions. In Section 3, using employing a combination of analytical and numerical methods, we compare the optimal price, output and profit under Cournot competition with those under Bertrand competition. Concluding remarks are given in Section 4.

2 $n$-Firm Oligopoly Models

We will assume consumer’s utility maximization in Section 2.1 to obtain a special demand function. In Section 2.2, the firm’s profit maximization will be considered under quantity (Cournot) competition, and in Section 2.3 we will derive the optimal prices, outputs and profits under price (Bertrand) competition.

2.1 Consumers

As in Singh and Vives (1984) and Häckner (2000), it is assumed that there is a continuum of consumers of the same type and the utility function of the representative consumer is given as

$$U(q,I) = \sum_{i=1}^{n} \alpha_i q_i - \frac{1}{2} \left( \sum_{i=1}^{n} q_i^2 + 2 \gamma \sum_{i \neq j}^{n} q_i q_j \right) - I,$$  

(1)

where $q = (q_i)$ is the quantity vector, $I = \sum_{i=1}^{n} p_i q_i$ with $p_i$ being the price of good $k$, $\alpha_i$ measures the quality of good $i$ and $\gamma \in [-1,1]$ measures the degree of relation between the goods: $\gamma > 0$, $\gamma < 0$ or $\gamma = 0$ imply that the goods are substitutes, complements or independent. Moreover, the goods are perfect substitutes if $\gamma = 1$ and perfect complements if $\gamma = -1$. In this study, we confine our analysis to the case in which the goods are imperfect substitutes or complements and are not independent, by assuming that $|\gamma| < 1$ and $\gamma \neq 0$.

The linear inverse demand function (or the price function) of good $k$ is obtained from the first-order condition of the interior optimal consumption of good $k$ and is given by

$$p_k = \alpha_k - q_k - \gamma \sum_{i \neq k}^{n} q_i \text{ for } k = 1, 2, ..., n,$$

(2)

where $n \geq 2$ is assumed. That is, the price vector is a linear function of the output vector:

$$p = \alpha - Bq,$$

(3)
where $p = (p_i)$, $\alpha = (\alpha_i)$ and $B = (B_{ij})$ with $B_{ii} = 1$ and $B_{ij} = \gamma$ for $i \neq j$. Assuming that $B$ is invertible\(^2\) and then solving (3) for $q$ yield the direct demand

$$q = B^{-1}(\alpha - p)$$

(4)

where the diagonal and the off-diagonal elements of $B^{-1}$ are, respectively,

$$\frac{1 + (n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)} \quad \text{and} \quad -\frac{\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)}.$$  

Hence the direct demand of good $k$, the $k^{th}$-component of $q$, is linear in the other firms’ prices and is given by

$$q_k = \frac{(1 + (n - 2)\gamma)(\alpha_k - p_k) - \gamma \sum_{i \neq k} (\alpha_i - p_i)}{(1 - \gamma)(1 + (n - 1)\gamma)}.$$  

(5)

Since Singh and Vives (1984) have already examined the duopoly case (i.e., $n = 2$), we will consider a more general case of $n > 2$ henceforth. For the sake of the later analysis, let us define the admissible region of $(\gamma, n)$ by $D_+$ or $D_-$ according to whether the goods are substitutes or complements;

$$D_+ = \{(\gamma, n) | 0 < \gamma < 1 \text{ and } 2 < n\}$$

and

$$D_- = \{(\gamma, n) | -1 < \gamma < 0 \text{ and } 2 < n\}.$$  

2.2 Quantity-adjusting firms

In Cournot competition, firm $k$ chooses a quantity $q_k$ of good $k$ to maximize its profit $\pi_k = (p_k - c_k)q_k$ subject to its price function (2), taking the other firms’ quantities given. We assume a linear cost function for each firm, so that the marginal cost $c_k$ is constant and non-negative. To avoid negative optimal production, we also assume that the net quality of good $k$, $\alpha_k - c_k$, is positive.

**Assumption 1.** $c_k \geq 0$ and $\alpha_k - c_k > 0$ for all $k$.

Assuming interior maximum and solving its first-order condition yield the best reply of firm $k$ as,

$$q_k = \frac{\alpha_k - c_k}{2} - \frac{\gamma}{2} \sum_{i \neq k} q_i \text{ for } k = 1, 2, ..., n.$$  

(6)

It can be easily checked that the second-order condition is certainly satisfied. The Cournot equilibrium output and price for firm $k$ are obtained by solving the following simultaneous equations for $k = 1, 2, ..., n$,

$$q_k + \frac{\gamma}{2} \sum_{i \neq k} q_i = \frac{\alpha_k - c_k}{2}.$$  

\(^2\)The $n$ by $n$ matrix $B$ is invertible if $\det B = (1 - \gamma)^n(1 + (n - 1)\gamma) \neq 0$. The inequality constraint $1 + (n - 1)\gamma > 0$ will be assumed in Assumption 2 below and is a guarantee of the invertibility of $B$.  


or in vector form,
\[ B^C q = A^C, \]
where \( A^C = (\alpha_i - c_i)/2 \) and \( B^C = (B^C_{ij}) \) with \( B^C_{ii} = 1 \) and \( B^C_{ij} = \gamma/2 \) for \( i \neq j \). Since \( B^C \) is invertible, the Cournot output vector is given by
\[ q^C = (B^C)^{-1} A^C, \]
where the diagonal and off-diagonal elements of \( (B^C)^{-1} \) are, respectively,
\[ \frac{2(2 + (n - 2)\gamma)}{(2 - \gamma)(2 + (n - 1)\gamma)} \quad \text{and} \quad \frac{2\gamma}{(2 - \gamma)(2 + (n - 1)\gamma)}. \]
Hence the Cournot equilibrium output of firm \( k \) is
\[ q^C_k = \frac{\alpha_k - c_k}{2 - \gamma} - \frac{\gamma}{(2 - \gamma)(2 + (n - 1)\gamma)} \sum_{i=1}^{n} (\alpha_i - c_i) \quad \text{(7)} \]
and the Cournot equilibrium price of firm \( k \) is
\[ p^C_k = \frac{\alpha_k + c_k - \gamma c_k}{2 - \gamma} - \frac{\gamma}{(2 - \gamma)(2 + (n - 1)\gamma)} \sum_{i=1}^{n} (\alpha_i - c_i). \quad \text{(8)} \]
Subtracting (7) from (8) yields \( p^C_k - c_k = q^C_k \) and then by substituting it into the profit function, the Cournot profit is obtained:
\[ \pi^C_k = (q^C_k)^2. \quad \text{(9)} \]
The relation \( p^C_k - c_k = q^C_k \) implies that the Cournot price is positive if the Cournot output is positive. Equation (7) implies that the Cournot output is always positive when \( \gamma < 0 \). It also implies that the Cournot output with \( \gamma > 0 \) is non-negative if
\[ z^C(\gamma, n) = \frac{2 + (n - 1)\gamma}{n\gamma} \quad \text{(10)} \]
and \( z^C(\gamma, n) \) is the ratio of the average net quality over the individual net quality of firm \( k \),
\[ \beta_k = \frac{1}{n} \sum_{i=1}^{n} (\alpha_i - c_i) \quad \text{(12)} \]
When \( \beta_k < 1 \), the individual net quality of firm \( k \) is larger than the average net quality. Firm \( k \) is called higher-qualified in this case. On the other hand, when \( \beta_k > 1 \), the individual net quality is less than the average net quality. Firm \( k \) is then called lower-qualified.
We now turn our attention to the stability of the Cournot output. To treat (6) as a dynamic system, each firm assumes to have belief that the other firms remain unchanged with their outputs from the previous period. Then the first-order conditions give rise to the time invariant linear dynamic equation of firm \( k \)
\[ q_k(t + 1) = \frac{\alpha_k - c_k}{2} - \gamma \sum_{i \neq k} q_i(t), \quad k = 1, 2, ..., n. \quad \text{(13)} \]
Substituting $q_k(t + 1)$ into (2), the price function of firm $k$, yields the price dynamic equation associated with the output dynamics:

$$p_k(t + 1) = \alpha_k - q_k(t + 1) - \gamma \sum_{i \neq k} a_i(t + 1), \quad k = 1, 2, ..., n. \quad (14)$$

Equations (13) and (14) imply that the dynamic behavior of $p_k(t + 1)$ is essentially the same as that of $q_k(t + 1)$. In other word, the Cournot price is stable (resp. unstable) if the Cournot output is stable (resp. unstable). Therefore it is enough for our purpose to draw our attention only to the stability of the Cournot output.

The coefficient matrix of system (13) is its Jacobian:

$$J_C = \begin{pmatrix} 0 & -\frac{\gamma}{2} & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & -\frac{\gamma}{2} \\ & & \ddots & \ddots & \ddots \\ -\frac{\gamma}{2} & -\frac{\gamma}{2} & & 0 \end{pmatrix}.$$ 

The corresponding characteristic equation reads

$$|J_C - \lambda I| = (-1)^n \left( \lambda - \frac{\gamma}{2} \right)^{n-1} \left( \lambda + \frac{(n-1)\gamma}{2} \right) = 0,$$

which indicates that there are $n - 1$ identical eigenvalues and one different eigenvalue. Without a loss of generality, the first $n - 1$ eigenvalues are assumed to be identical,

$$\lambda_1^C = \lambda_2^C = \ldots = \lambda_{n-1}^C = \frac{\gamma}{2} \quad \text{and} \quad \lambda_n^C = -\frac{(n-1)\gamma}{2}.$$ 

Since $|\gamma| < 1$ is assumed, the first $n - 1$ eigenvalues are less than unity in absolute value. Hence stability of the Cournot output depends on the absolute value of $\lambda_n^C$. We have $|\lambda_2^C| = \left| -\frac{\gamma}{2} \right| < 1$ for $n = 2$ and $|\lambda_3^C| = |\gamma| < 1$ for $n = 3$. Thus the Cournot output is stable in case of duopoly or triopoly. If $|\lambda_n^C|$ is greater than unity for $n > 3$, then the Cournot output explosively oscillates. Solving $|\lambda_n^C| < 1$ presents the stability conditions of the Cournot output:

$$n < 1 + \frac{2}{\gamma} \quad \text{if} \quad \gamma > 0 \quad \text{and} \quad n < 1 - \frac{2}{\gamma} \quad \text{if} \quad \gamma < 0.$$ 

In summary: Cournot equilibrium can be locally unstable when the number of firms becomes more than three. The more precise stability results concerning the Cournot output and price are given in the following theorem:

**Theorem 1** Under Cournot competition with $n > 2$, (i) the Cournot output and price are stable for $(\gamma, n) \in R_n^S$ and unstable for $(\gamma, n) \in R_n^U = D(+) \setminus R_n^S$ if the goods are substitutes; (ii) they are stable for $(\gamma, n) \in R_n^C$ and unstable for $(\gamma, n) \in R_n^U = D(-) \setminus R_n^C$ if the goods are complements where the stability regions are, respectively, defined by

$$R_n^S = \{(\gamma, n) \in D(+) \mid n < 1 + \frac{2}{\gamma}\} \quad \text{and} \quad R_n^C = \{(\gamma, n) \in D(-) \mid n < 1 - \frac{2}{\gamma}\},$$
and the instability regions, $R^C_U$ and $R^C_w$, are the complements of the stability regions.

## 2.3 Price-adjusting firms

In Bertrand competition, firm $k$ chooses the price of good $k$ to maximize the profit $\pi_k = (p_k - c_k)q_k$ subject to its direct demand (5), taking the other firms’ prices given. Solving the first-order condition yields the best reply of firm $k$,

$$p_k = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n - 2)\gamma]} \sum_{i \neq k}^{n} (\alpha_i - p_i), \text{ for } k = 1, 2, ..., n. \quad (15)$$

The second-order condition for an interior optimum solution is

$$\frac{\partial^2 \pi_k}{\partial p_k^2} = -\frac{2(1 + (n - 2)\gamma)}{(1 - \gamma)(1 + (n - 1)\gamma)} < 0, \quad (16)$$

where the direction of inequality depends on the parameter configuration.\(^3\) For $(\gamma, n) \in D_{(+)}$, we see that (16) is always satisfied. On the other hand, for $(\gamma, n) \in D_{(-)}$, we need additional condition to fulfill the second-order condition. Since

$$1 + (n - 1)\gamma < 1 + (n - 2)\gamma,$$

for $\gamma < 0$, the required condition is either $0 < 1 + (n - 1)\gamma$ or $1 + (n - 2)\gamma < 0$.

As in Häckner (2000), we make the following assumption:

**Assumption 2.** $1 + (n - 1)\gamma > 0$ when $\gamma < 0$.

The Bertrand equilibrium prices are obtained by solving the simultaneous equations for $k = 1, 2, ..., n,$

$$p_k = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n - 2)\gamma]} \sum_{i \neq k}^{n} p_i = \frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n - 2)\gamma]} \sum_{i \neq k}^{n} \alpha_i$$

for unknown $p_k (k = 1, 2, ..., n)$, or in vector form

$$B^p p = A^B,$$

where $A^B = \left(\frac{\alpha_k + c_k}{2} - \frac{\gamma}{2[1 + (n - 2)\gamma]} \sum_{i \neq k}^{n} \alpha_i \right)$ and $B^B = (B^B_{ij})$ with $B^B_{ii} = 1$ and $B^B_{ij} = -\frac{\gamma}{2[1 + (n - 2)\gamma]}$ for $i \neq j$. Since $B^B$ is invertible, the solution is

$$p = (B^B)^{-1} A^B,$$

where the diagonal and off-diagonal elements of $(B^B)^{-1}$ are, respectively,

$$\frac{2(1 + (n - 2)\gamma)(2 + (n - 2)\gamma)}{(2 + (n - 3)\gamma)(2 + (2n - 3)\gamma)} \quad \text{and} \quad \frac{2\gamma(1 + (n - 2)\gamma)}{(2 + (n - 3)\gamma)(2 + (2n - 3)\gamma)}.$$

\(^3\)Note that inequality (16) is always fulfilled for $n = 2$. 

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Hence, the Bertrand equilibrium price and output of firm $k$ are given by

$$p_k^B = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)(\alpha_k+c_k)-\gamma(1+(n-2)\gamma)\sum_{i\neq k}\alpha_i-c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)} \quad (17)$$

and

$$q_k^B = \frac{1 + (n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)}(p_k^B - c_k) \quad (18)$$

with

$$p_k^B - c_k = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)(\alpha_k-c_k)-\gamma(1+(n-2)\gamma)\sum_{i\neq k}\alpha_i-c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)}. \quad (19)$$

Due to (18), the Bertrand profit of firm $k$ becomes

$$\pi_k^B = \frac{(1 - \gamma)(1 + (n - 1)\gamma)}{1 + (n - 2)\gamma}(q_k^B)^2. \quad (20)$$

Equation (18) implies that the Bertrand output is positive if $p_k^B - c_k$ is positive. Under Assumption 2, equation (19) implies that $p_k^B - c_k$ is always positive if $\gamma < 0$. It also implies that $p_k^B - c_k$ with $\gamma > 0$ is non-negative if

$$z^B(\gamma, n) \geq \beta_k \quad (21)$$

where

$$z^B(\gamma, n) = \frac{(2 + (n - 3)\gamma)(1 + (n - 1)\gamma)}{(1 + (n - 2)\gamma)n\gamma} \quad (22)$$

and $\beta_k$ is defined by (12).

To consider stability of the Bertrand price, we assume naive expectations on price formation and obtain the following time-invariant difference equations for price dynamics:

$$p_k(t+1) = \frac{\alpha_k + c_k - \gamma}{2[1 + (n - 2)\gamma]} \sum_{i\neq k} [\alpha_i - p_i(t)]. \quad (23)$$

Similarly to the Cournot competition, we can also obtain the output difference equation under Bertrand competition by substituting $p_k(t+1)$ into the direct demand function (5):

$$q_k(t+1) = \frac{(1 + (n - 2)\gamma)(\alpha_k - p_k(t+1)) - \gamma \sum_{i\neq k} (\alpha_i - p_i(t+1))}{(1 - \gamma)(1 + (n - 1)\gamma)}. \quad (24)$$

It is clear from (23) and (24) that the output dynamics is synchronized with the price dynamics. The coefficient matrix of this price adjusting system is

$$J_B = \begin{pmatrix} 0 & \gamma & \gamma \\ \gamma & 2[1 + (n - 2)\gamma] & 2[1 + (n - 2)\gamma] \\ 2[1 + (n - 2)\gamma] & 0 & 2[1 + (n - 2)\gamma] \end{pmatrix}.$$
Its characteristic equation is

$$[J_B - \lambda I] = (-1)^n \left( \lambda + \frac{\gamma}{2[1 + (n - 2)\gamma]} \right)^{n-1} \left( \lambda - \frac{(n-1)\gamma}{2[1 + (n - 2)\gamma]} \right) = 0,$$

and by assuming that the first $n - 1$ eigenvalues are equal,

$$\lambda_1^B = \lambda_2^B = \ldots = \lambda_{n-1}^B = -\frac{\gamma}{2[1 + (n - 2)\gamma]} \quad \text{and} \quad \lambda_n^B = \frac{(n-1)\gamma}{2[1 + (n - 2)\gamma]}.$$

When $\gamma > 0$ and $n > 2$, we have $|\lambda_k^B| < 1$ for $k = 1, 2, \ldots, n$. That is, the Bertrand price is asymptotically locally stable in $D_{(+)}$. On the other hand, when $\gamma < 0$, the condition $0 > \lambda_n^B > -1$ can be rewritten as

$$n < \frac{5}{3} - \frac{2}{3\gamma},$$

under which $0 < \lambda_k^B < 1$ also holds for $k = 1, 2, \ldots, n - 1$. Since the Bertrand competition synchronizes output dynamics with price dynamics, the stability of the Bertrand price and output are summarized as follows:

**Theorem 2** Under Bertrand competition with $n > 2$, (i) the Bertrand price and output are stable if the goods are substitute; (ii) if the goods are complements, then they are stable for $(\gamma, n) \in R_s^B$ and unstable for $(\gamma, n) \in R_u^B = D^B \setminus R_s^B$ where $D^B$ is the feasible region under Assumption 2,

$$D^B = \{ (\gamma, n) \in D_{(-)} \mid 0 < 1 + (n - 1)\gamma \},$$

$R_s^B$ is the stable region,

$$R_s^B = \{ (\gamma, n) \in D^B \mid n < \frac{5}{3} - \frac{2}{3\gamma} \}$$

and $R_u^B$ is the unstable region, which is the complement of the stable region.

Theorems 1 and 2 consider stability of the Cournot output and the Bertrand price as well as stability of the Cournot price and the Bertrand output through the difference equations (14) and (24). Graphical explanations of Theorems 1 and 2 are given in Figure 1. In the first quadrant where $\gamma > 0$, the admissible region $D_{(+)}$ is divided into two parts by the neutral stability locus of the Cournot output $\lambda_C^U = -1$; the light-gray region $R_C^U$ below the locus and the dark-gray region $R_C^L$ above. The Cournot output is stable in the former and unstable in the latter while the Bertrand price is stable in both regions. In the second quadrant where $\gamma < 0$, the admissible region of the Bertrand price is reduced to $D^B$ from $D_{(-)}$ by Assumption 2. The neutral stability locus of the Bertrand price $\lambda_n^B = -1$ cuts across the locus of $1 + (n - 1)\gamma = 0$ from left to right at point $(-1/2, 3)$ and divides the region $D^B$ into two parts: the light-gray region $R_u^B$ and the dark-gray region $R_s^B$. The Bertrand price is stable in the former and unstable in the latter. In comparing the Cournot and the Bertrand strategies, we should confine our analysis to the parametric region in which both equilibria are feasible, otherwise the comparison has no economic meanings. Consequently
two things are certain. One is that we can ignore the white region of the second quadrant in all further discussions as Assumption 2 is violated there. The other is that the Cournot output is always stable in $D^B$ since the stable region of the Cournot output is under the locus of $\lambda_n^C = 1$ and is larger than $D^B$.

Figure 1. Stable and unstable regions

Theocharis (1960) studies stability of discrete dynamic evolution of the Cournot output under naive expectation when the goods are perfect substitutes (i.e., no product differentiation) and demonstrates that the Cournot output is asymptotically stable if and only if the number of firms is equal to two. In our analysis, it is easy to see that for $n = 3$, the Cournot output is monotonically stable, and for $n > 3$, it is unstable. Theorems 1 and 2 extend Theocharis’ classical result and assert that the Cournot output as well as the Bertrand price can be unstable when the number of firms is greater than three and the goods are differentiated. The main point is that Cournot output can be unstable only when the goods are substitutes and so can be Bertrand price only when the goods are complements.

3 Optimal Strategy Comparison

We call the optimal price, output and profit under Cournot competition Cournot strategy and those under Bertrand competition Bertrand strategy. In this section we will compare Cournot strategy with Bertrand strategy to examine which strategy is more preferable when the number of the firms becomes more than three.

Assuming $n > 2$ and subtracting (17) from (8) yield a price differential

$$p_C^k - p_B^k = \frac{(\alpha_k - c_k)(n - 1)(1 - \gamma)(2 - \gamma)(2 + (2n - 3)\gamma) z^p(\gamma, n)}{z^p(\gamma, n) - \beta_k} ,$$

Although we do not refer to dynamics of Cournot price and Bertrand output, the dynamic equations (14) and (24) imply that Cournot price exhibits the same movement as Cournot output and so does Bertrand output as Bertrand price.
where 
\[ z^P(\gamma, n) = \frac{(2 + (n - 1)\gamma)(2 + (n - 3)\gamma)}{(n - 2)n\gamma^2}. \]

The first two factors multiplying the parenthesized term on the right hand side of (25) are positive implying that
\[ \text{sign} \left[ p_k^C - p_k^B \right] = \text{sign} \left[ z^P(\gamma, n) - \beta_k \right]. \] (26)

Subtracting (18) from (7) yields an output differential,
\[ q_k^C - q_k^B = \frac{(\alpha_k - c_k)(n - 1)\gamma^2}{(2 - \gamma)(1 - \gamma)(2 + (2n - 3)\gamma)} \frac{1}{z^Q(\gamma, n)} \left( \beta_k - z^Q(\gamma, n) \right) \] (27)

where
\[ z^Q(\gamma, n) = \frac{(2 + (n - 3)\gamma)(1 + (n - 1)\gamma)(2 + (n - 1)\gamma)}{n\gamma(4 + 5(n - 2)\gamma + (n^2 - 5n + 5)\gamma^2)}. \]

The first factor on the right hand side of (27) is positive while the second one (i.e., the reciprocal of \( z^Q(\gamma, n) \)) is ambiguous: it is positive when the goods are substitutes and negative when the goods are complements. The sign of the output differential is determined by the simplified expression,
\[ \text{sign} \left[ q_k^C - q_k^B \right] = \text{sign} \left[ \gamma \left( \beta_k - z^Q(\gamma, n) \right) \right]. \] (28)

Finally, dividing (9) by (20) gives a profit ratio,
\[ \frac{\pi_k^C}{\pi_k^B} = 1 + \frac{(n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)} \left( \frac{q_k^C}{q_k^B} \right)^2. \]

Since the first factor on the right hand side is positive and greater than unity, we have \( \pi_k^C > \pi_k^B \) if \( q_k^C > q_k^B \). In order to find a more general condition determining whether the profit ratio is greater or less than unity, we substitute (18) and (7) into the last expression to have
\[ \frac{\pi_k^C}{\pi_k^B} = G(\beta_k) \]
with
\[ G(\beta_k) = B(\gamma, n) \left( A(\gamma, n) \frac{z^C(\gamma, n) - \beta_k}{z^B(\gamma, n) - \beta_k} \right)^2, \] (29)

where
\[ A(\gamma, n) = \frac{(1 - \gamma)(1 + (n - 1)\gamma)(2 + (2n - 3)\gamma)(2 + (n - 3)\gamma)}{(2 - \gamma)(2 + (n - 1)\gamma)(1 + (n - 2)\gamma)^2} > 0 \]
and
\[ B(\gamma, n) = \frac{1 + (n - 2)\gamma}{(1 - \gamma)(1 + (n - 1)\gamma)} > 0. \]

When the net quality of firm \( k \) is equal to the average net quality offered by all firms, the profit ratio is
\[ G(1) = \frac{(2 + (n - 3)\gamma)^2(1 + (n - 1)\gamma)}{(1 - \gamma)(1 + (n - 2)\gamma)(2 + (n - 1)\gamma)^2}. \]
The difference of the denominator and the numerator of $G(1)$ is
\[(n - 1)^2(2 + (n - 2)\gamma)\gamma^3,
\]
which then implies that
\[G(1) > 1 \text{ if } \gamma > 0 \text{ and } G(1) < 1 \text{ if } \gamma < 0, \quad (30)
\]
Differentiating $G(\beta_k)$ with respect to $\beta_k$ gives, after arranging terms,
\[
\frac{dG(\beta_k)}{d\beta_k} = \frac{2A(\gamma, n)^2B(\gamma, n)(z^C(\gamma, n) - z^B(\gamma, n))(z^C(\gamma, n) - \beta_k)}{(z^B(\gamma, n) - \beta_k)^2}, \quad (31)
\]
Noticing that $z^C(\gamma, n) - z^B(\gamma, n) > 0$, $z^C(\gamma, n) - \beta_k > 0$ and $z^B(\gamma, n) - \beta_k > 0$ when $\gamma > 0$ and $z^C(\gamma, n) - z^B(\gamma, n) < 0$, $z^C(\gamma, n) - \beta_k < 0$ and $z^B(\gamma, n) - \beta_k < 0$ when $\gamma < 0$, we find that the sign of the derivative of $G(\beta_k)$ is positive when the goods are substitutes and negative when complements:
\[
\frac{dG(\beta_k)}{d\beta_k} > 0 \text{ when } \gamma > 0 \text{ and } \frac{dG(\beta_k)}{d\beta_k} < 0 \text{ when } \gamma < 0. \quad (32)
\]

3.1 Duopoly Case: $n = 2$
As a benchmark, we consider the duopoly case and confirm the Singh-Vives results in our framework. Substituting $n = 2$ into (25), (27) and (29) yields
\[p^C_k - p^B_k = \frac{(\alpha_k - c_k)\gamma^2}{4 - \gamma^2}, \quad (33)
\]
\[q^C_k - q^B_k = \frac{\gamma^2}{(1 - \gamma^2)(4 - \gamma^2)} \left\{ 2(\alpha_k - c_k)\gamma \left[ \beta_k - \frac{1 + \gamma}{2\gamma} \right] \right\}, \quad (34)
\]
and
\[G(\beta_k) = (1 - \gamma^2) \left\{ \frac{z^C - \beta_k}{z^B - \beta_k} \right\}^2, \quad (35)
\]
where
\[z^C = \frac{2 + \gamma}{2\gamma} \text{ and } z^B = \frac{(2 - \gamma)(1 + \gamma)}{2\gamma}.
\]
Given $|\gamma| < 1$, it is fairly straightforward that $p^C_k > p^B_k$ always and $q^C_k < q^B_k$ when $\gamma < 0$. To determine the sign of the output difference in case of $\gamma > 0$, we return to the direct demand (5) and consider consequences of the assumption $\alpha_i - \gamma\alpha_j > 0$ for $i \neq j$, which is implicitly imposed to guarantee that the independent or non-induced demand for $p_i = 0$, $i = 1, 2$ is positive when the goods are substitutes. This assumption can be rewritten as
\[\alpha_k - \gamma\alpha_j = 2\alpha_k\gamma \left( \frac{1 + \gamma}{2\gamma} - z_k \right) > 0
\]
with
\[z_k = \frac{1}{2} \sum_{i=1}^{2} \frac{\alpha_i}{\alpha_k}
\]
being the ratio of the average quality of two firms over the individual quality of firm \( k \). Since \( \gamma > 0 \) and \( \alpha_k > 0 \), this inequality indicates that an upper bound is imposed on \( z_k \),

\[
z_k < \frac{1 + \gamma}{2\gamma}.
\]

Furthermore, \( \alpha_i - \gamma \alpha_k > 0 \) can be rewritten as

\[
\frac{\alpha_i}{\alpha_k} > \frac{2\gamma}{1 + \gamma} z_k,
\]

which is substituted into the definition of \( z_k \) to have

\[
z_k > \frac{1 + \gamma}{2}.
\]

If \( c_k = 0 \), then it is apparent from the definitions that \( \beta_k = z_k \). In the future discussions, we retain \( c_i > 0 \) and make the following assumption,

\[
\frac{c_1}{\alpha_1} = \frac{c_2}{\alpha_2},
\]

under which, it is not difficult to show that \( \beta_k = z_k \). Then \( \beta_k \) is bounded from above and also from below

\[
\frac{1 + \gamma}{2} < \beta_k < \frac{1 + \gamma}{2\gamma}
\]

and

\[
2(\alpha_k - c_k)\gamma \left( \frac{1 + \gamma}{2\gamma} - \beta_k \right) > 0.
\]

With the last inequality, the output difference (34) implies that \( q_k^C < q_k^B \) in case of \( \gamma > 0 \). Hence we have \( q_k^C < q_k^B \) always regardless of whether the goods are substitutes or complements.

Substituting \( n = 2 \) into (31) gives

\[
\frac{dG(\beta_k)}{d\beta_k} = \gamma z_k^C - \beta_k \geq 0 \text{ if } \gamma \geq 0.
\]

The minimum value of \( \beta_k \) is \((1 + \gamma)/2\) when \( \gamma > 0 \) and \( 1/2 \) when \( \gamma < 0 \), which is substituted into the profit ratio (35) to obtain

\[
G \left( \frac{1 + \gamma}{2} \right) = \frac{(2 - \gamma^2)^2}{4 - \gamma^2} > 1 \text{ and } G \left( \frac{1}{2} \right) = \frac{4 - \gamma^2}{(2 - \gamma^2)^2} < 1.
\]

The value of \( G(\beta_k) \) increases in \( \beta_k \) and is greater than unity for the minimum value of \( \beta_k \) when \( \gamma > 0 \) whereas it decreases and is less than unity for the minimum value of \( \beta_k \) when \( \gamma < 0 \). Hence we obtain that

\[
\pi_k^C > \pi_k^B \text{ if } \gamma > 0 \text{ and } \pi_k^C < \pi_k^B \text{ if } \gamma < 0.
\]

We have therefore confirmed the Singh-Vives results, (i-SV), (ii-SV) and (iii-SV), mentioned in the Introduction and now we will proceed to the \( n \)-firm case in order to examine the effects of the increasing number of firms on these results.
3.2 The goods are substitutes, $\gamma > 0$

We first assume that firm $k$ is higher-qualified (i.e., $\beta_k \leq 1$). If $\gamma > 0$, then $z^F(\gamma, n) > z^C(\gamma, n) > z^B(\gamma, n) > z^Q(\gamma, n) > 1$ and thus

$$z^Q(\gamma, n) > \beta_k.$$  

With this inequality, equations (11), (21), (25) and (28) imply the following three results: (i) $q_k^C$ and $p_k^C$ are positive; (ii) $q_k^B$ and $p_k^B$ are positive; (iii) $p_k^C > p_k^B$ and $q_k^C < q_k^B$. Before examining the profit ratio, we assume, as in the duopoly case, that the non-induced demand of (5) is positive:

**Assumption 3.** $(1 + (n - 2)\gamma) \alpha_k - \gamma \sum_{i \neq k} \alpha_i > 0$.

This assumption can be rewritten as

$$\alpha_k n \gamma \left( \frac{1 + (n - 1)\gamma}{n \gamma} - z_k \right) > 0,$$

where

$$z_k = \frac{1}{n} \sum_{i \neq k} \frac{\alpha_i}{\alpha_k}$$

is the ratio of the average quality over the individual quality of firm $k$. The above inequality implies that Assumption 3 imposes an upper bound on $z_k$,

$$z_k < \frac{1 + (n - 1)\gamma}{n \gamma}.$$  

The same assumption for firm $j$ ($\neq k$) can be converted into

$$\alpha_j > \frac{n \gamma}{1 + (n - 1)\gamma} z_k \alpha_k.$$

This inequality is substituted into the definition of $z_k$ to obtain the lower bound of $z_k$,

$$z_k > \frac{1 + (n - 1)\gamma}{n}.$$  

That is, Assumption 3 restricts the value of $z_k$ into an interval by imposing upper and lower bounds. To simplify the relation between $z_k$ and $\beta_k$, we make one more assumption that the ratio of the unit cost over the quality of firm $k$ is identical with the ratio of the average cost over the average quality in the market:

**Assumption 4.** $\frac{c_k}{\alpha_k} = \frac{\sum_{i=1}^{n} c_i / n}{\sum_{i=1}^{n} \alpha_i / n}$.

Under Assumptions 3 and 4, the net quality ratio of firm $k$ has the upper and lower bounds,

$$\beta_k^u = \frac{1 + (n - 1)\gamma}{n} < 1 \quad \text{and} \quad \beta_k^l = \frac{1 + (n - 1)\gamma}{n \gamma} > 1$$

and satisfies inequality.
\[(\alpha_k - c_k)n\gamma \left( \frac{1 + (n-1)\gamma}{n\gamma} - \beta_k \right) > 0.\]

Substituting \(\beta_k^m\) and \(\beta_k^M\) into (29) gives

\[G(\beta_k^m) < 1 \text{ and } G(\beta_k^M) > 1,\]

We will next numerically examine the cases where \(G(\beta_k^m)\) is greater or less than unity. If \(G(\beta_k^m) < 1\), then there is a threshold value \(\bar{\beta}_k\), making \(G(\beta_k) = 1\), since \(G(\beta_k^m) > 1\) and \(G'(\beta_k) > 0\). Solving \(G(\beta_k) = 1\) gives an explicit form of \(\bar{\beta}_k\),

\[\bar{\beta}_k(\gamma, n) = \frac{zB - A^2CzC - A(zC - zB)\sqrt{B}}{1 - A^2B},\]

where the dependency of each term on \(\gamma\) and \(n\) is omitted for the sake of notational simplicity. Given \(\beta_k\), \(\gamma\) and \(n\), we have the following results on the profit differences:

- if \(\beta_k < \bar{\beta}_k(\gamma, n)\), then \(G(\beta_k) < 1\) implying \(\pi_k^C < \pi_k^B\)
- if \(\beta_k > \bar{\beta}_k(\gamma, n)\), then \(G(\beta_k) > 1\) implying \(\pi_k^C > \pi_k^B\).

The net quality ratio \(\beta_k\) and its minimum value \(\beta_k^m\) are assumed to be less than unity. Thus depending on the configuration of \((\gamma, n)\), the locus of \(\beta_k^m = \beta_k\) divides the admissible region \(D_{(+)}\) into two parts. One is a region with \(\beta_k^m > \beta_k\) in which Assumption 3 is violated and the other is a region with \(\beta_k^m < \beta_k\). The former region is discarded. The latter region is further divided by the locus of \(G(\beta_k^m) = 1\) into two parts: one region with \(G(\beta_k^m) < 1\) and the other with \(G(\beta_k^m) > 1\). Since \(\beta_k^m < \beta_k\) in this latter region, \(G(\beta_k^m) > 1\) and \(G'(\beta_k) > 0\) lead to \(G(\beta_k) > 1\) implying that \(\pi_k^C > \pi_k^B\). Finally the locus of \(\beta_k(\gamma, n) = \beta_k\) divides the former region with \(G(\beta_k^m) < 1\) into two parts. We have \(\pi_k^C > \pi_k^B\) in one region with \(\beta_k(\gamma, n) < \beta_k\) where \(G(\beta_k) = 1 < G(\beta_k)\) and \(\pi_k^C < \pi_k^B\) in the other region with \(\beta_k(\gamma, n) > \beta_k\) where \(G(\beta_k) = 1 > G(\beta_k)\). A graphical representation of dividing \(D_{(+)}\) is given in Figure 2 in which \(\beta_k = 1/2\). There the downward-sloping hyperbola is the neutral stability locus. The positive sloping curve is the \(\beta_k^m = \beta_k\) locus. Assumption 3 is violated in the white region in the right side of this locus. The U-shaped curve is the \(G(\beta_k^m) = 1\) locus and the equal-profit locus is negative-sloping and divides the region with conditions \(G(\beta_k^m) < 1\) and \(\beta_k^m < \beta_k\) into two parts. In the horizontally-striped region, \(\pi_k^C < \pi_k^B\) and the inequality is reversed in the non-striped region. Notice the two important issues. The first issue is that \(\pi_k^C < \pi_k^B\) holds in the horizontally-striped region. Emergence of dominant Bertrand profit over Cournot profit has been already pointed out by Häckner (2000) in his Proposition 2(ii). We can confirm it and further construct a set of pairs \((\gamma, n)\) for which it holds under Assumptions 3 and 4. The second issue is that the Cournot output and price are locally unstable whenever \(\pi_k^C < \pi_k^B\) since the horizontally-striped region is located within the unstable region, which is the dark-gray domain surrounded by the two loci of \(\lambda_n^C = -1\) and \(\beta_k^m = \beta_k\). In summary, we arrive at the following conclusions when \(\gamma > 0\) and \(\beta_k \leq 1\):
Theorem 3 When $\beta_k$, $\gamma$ and $n$ are given such that firm $k$ is higher-qualified and $\beta^m_k < \beta_k$, then (i) firm $k$ charges a higher price and produces smaller output under Cournot competition than under Bertrand competition; (ii) its Cournot profit is higher than its Bertrand profit for $\beta_k > \beta_k(\gamma, n)$ and $G(\beta^m_k) < 1$ whereas the profit dominance is reversed otherwise.

Next, firm $k$ is assumed to be lower-qualified (i.e., $\beta_k > 1$). In order to make any comparison meaningful, we have to find out under what parametric configurations of $\gamma$ and $n$ Assumption 3 is satisfied. Since $\beta^m_k < \beta_k$ is satisfied by assumption, $\beta_k$ should be chosen to be less than its upper bound. The locus of $\beta^M_k = \beta_k$ divides the admissible region $D(+)_{k}$ into two parts, $R_+ = \{ (\gamma, n) \in D(+) | \beta^M_k \geq \beta_k \}$ and $R_- = \{ (\gamma, n) \in D(+) | \beta^M_k < \beta_k \}$.

For $(\gamma, n) \in R_-$, Assumption 3 is violated. So we eliminate this region from all further considerations and confine our attention to $R_+$. When $\gamma > 0$, we have the following orderings

$$z^D(\gamma, n) > z^C(\gamma, n) > z^B(\gamma, n) > 1 \quad \text{and} \quad z^B(\gamma, n) > \beta^M_k > z^Q(\gamma, n).$$

Consequently, $z^D(\gamma, n) > \beta_k$ in $R_+$ and then equation (25) indicates that $p^C_k > p^B_k$ always in $R_+$. Since $\beta_k > 1$, $G(1) > 1$ and $G'(\beta_k) > 0$ lead to $G(\beta_k) > 1$, so equation (29) indicates that $\pi^C_k > \pi^B_k$ always in $R_+$.

The indeterminacy of the relative magnitude between $z^Q(\gamma, n)$ and $\beta_k$ implies that the equal-product locus of $z^Q(\gamma, n) = \beta_k$ divides $R_+$ into two parts. In a part with $z^Q(\gamma, n) < \beta_k$, the Cournot output is larger than the Bertrand output, according to equation (27). Notice that the case of $q^C_k > q^B_k$ does not
emerge in duopolies and its possibility is not examined in Häckner (2000). The $M_k = \frac{k}{\lambda} = C_n = 1$ locus crosses the $\lambda^C_n = -1$ locus at point $(\hat{\gamma}, \hat{n})$ with

$$\hat{\gamma} = \frac{3}{\beta_k} - 2 \text{ and } \hat{n} = 1 + \frac{2}{\hat{\gamma}}.$$ 

Here $\hat{\gamma}$ is positive for $\beta_k \in (1, 3/2)$ and decreases monotonically form 1 to 0 as $\beta_k$ increases from 1 to 3/2. Accordingly, $\hat{n}$ increases from 3 to infinity. Graphically this means that the intersection $(\hat{\gamma}, \hat{n})$ moves upwards along the neutral stability locus since the upper bound curve shifts leftward in $R_+$ as $\beta_k$ increases. It also means that the two curves do not cross for $\beta_k > 3/2$ and the upper bound curve is within the stable region. Summarizing these observations, we possibly obtain $q^C_k > q^B_k$ for $n \geq 3$ when $\gamma > 0$ and $\beta_k > 1$. Two examples of the division of $R_+$ are given in Figure 3. In Figure 3(A) where $\beta_k = 1.15$, $q^C_k > q^B_k$ in the horizontally-striped and hatched regions. Furthermore $q^C_k$ is unstable in the horizontally-striped region and stable in the hatched region. In Figure 3(B) where $\beta_k = 3/2$, $q^C_k > q^B_k$ in the horizontally-striped region and $q^C_k < q^B_k$ in the non-striped region and $q^C_k$ is stable in both regions. Since $\beta_k$ is larger than its upper bound, $M_k$, in the white regions right to the $M_k = \beta_k$ locus, we discard them. Comparing Figure 3(A) with Figure 3(B), it can be seen that the whole horizontally-striped region is inside the stable region for $\beta_k > 3/2$ and some part of the region is outside the stable region for $\beta_k < 3/2$. That is, $q^C_k$ can be unstable for a relative large value of $n$ and $\beta_k < 3/2$ whereas it is always stable for $\beta_k \geq 3/2$. We summarize these results as follows:

**Theorem 4** When $\beta_k$, $\gamma$ and $n$ are given such that firm $k$ is lower-qualified and $\beta_k < \beta^M_k$, then (i) firm $k$ charges a higher price and earns a larger profit under Cournot competition than under Bertrand competition; (ii) its Cournot output is smaller than its Bertrand output for $\beta_k > z^Q(\gamma, n)$ and the relation is reversed for $\beta_k < z^Q(\gamma, n)$.

![Figure 3](image_url)

Figure 3. Stable and unstable regions in $D_{(+)}$ when $\beta_k > 1$
3.3 The goods are complements, $\gamma < 0$

When $\gamma < 0$, equation (28) implies $q_k^C < q_k^B$ always as $z^Q(\gamma, n) < 0$ for any $\gamma$ and $n$ in $D^B$ regardless of whether $\beta_k$ is greater or less than unity. It should be noticed that the non-induced demand is always positive and Assumption 3 is not necessary. However, the lower bound of $\beta_k$ is defined to be $1/n$ when the net qualities of any other firms are zero.

The profit ratio for $\beta_k = 1/n$ is reduced to

$$G\left(\frac{1}{n}\right) = A(\gamma, n)^2 B(\gamma, n) \left(\frac{(1 + (n - 2)\gamma)(2 + (n - 2)\gamma)}{2 + 3(n - 2)\gamma + (n^2 - 5n + 5)\gamma^2}\right)^2.$$ 

Although it is indeterminate in general whether $G(1/n)$ is greater or less than unity,\(^5\) it can be shown that

$$G\left(\frac{1}{n}\right) < 1 \text{ for } \gamma \in (\gamma_0, 0) \text{ when } n = 8$$

and

$$G\left(\frac{1}{n}\right) > 1 \text{ for } \gamma \in (\gamma_1, \gamma_2) \text{ when } n = 9$$

where $\gamma_0 = 1/(n - 1)$, $\gamma_i \in (\gamma_0, 0)$ for $i = 1, 2$ are the solutions of equation $G(1/n) = 1$. Hence there is a threshold value $\bar{n} = (8, 9)$ of $n$ such that $G(1/n) = 1$ has a unique solution $\hat{\gamma} \in (\gamma_0, 0)$. It is numerically obtained that $\hat{\gamma} \approx -0.133$ and $\bar{n} = 8.16$. Since it is indeterminate whether $G(1/n)$ is greater or less than unity, the $G(1/n) = 1$ locus divides the region $D^B$ into two parts, one with $G(1/n) < 1$ and the other with $G(1/n) > 1$, which is above the $n = \bar{n}$ line.

If $n < n_k$, then $\beta_k^m (= 1/n) > \beta_k$. This inequality violates the assumption that the net quality $\beta_k$ is greater or equal to its lower bound, $\beta_k^m$. Thus this case is eliminated from considerations. Having $n \geq n_k$, we should still distinguish between the following three cases,

$$n_k < n \leq \bar{n}, \quad n_k \leq \bar{n} \leq n \text{ and } \bar{n} < n_k < n.$$ 

We start with the first case, $n_k < n \leq \bar{n}$. Condition $n \leq \bar{n}$ and $G' < 0$ imply that $G(1/n) \leq G(1/\bar{n})$ for $\gamma \in (\gamma_0, 0)$. It has been shown that $G(1/\bar{n}) \leq 1$ for $\gamma \in (\gamma_0, 0)$, $\beta_k \geq 1/n (= \beta_k^m)$ implies that $G(\beta_k) \leq G(1/n) \leq 1$. Hence our first result on the profit difference is that

$$\pi_k^C \leq \pi_k^B \text{ if } n_k < n \leq \bar{n}.$$ 

Returning to the definitions of $\tilde{\beta}_k$ and $\bar{n}$, we have

$$\tilde{\beta}_k(\gamma, \bar{n}) = \frac{1}{n}.$$ 

We denote this threshold value of $\beta_k$ by $\tilde{\beta} = \tilde{\beta}_k(\gamma, \bar{n})$ or $\tilde{\beta} = 1/\bar{n}$. Simple calculation shows that $\tilde{\beta} \approx 0.123$. In Figure 4 below, the graphs of $\tilde{\beta}(\gamma, n)$ with changing values of $n$ are depicted against $\gamma$. The left most graph is for

\(^5\)It is shown that $\lim_{\gamma \to 0} G(1/n) = 1$ and $\lim_{\gamma \to 1/(1-n)} G(1/n) = 0$.
\( n = \hat{n}(\approx 8.16) \) and the next to the right graph is for \( n = 9 \). As the number of \( n \) increases from 9 to 33 with two increments, the graph moves rightward accordingly. The right most graph is for \( n = 33 \). It can be observed that the maximum value of \( \tilde{\beta}(\gamma, n) \) decreases with increasing number of \( n \). In other words, the maximum value of \( \tilde{\beta}(\gamma, n) \) is \( \hat{\beta} \) for \( n \geq \hat{n} \),

\[
\max_{n \geq \hat{n}, \gamma_0 < \gamma < 0} \tilde{\beta}(\gamma, n) = \hat{\beta}. \tag{36}
\]

If \( n_k \leq \hat{n} \leq n \), then \( G(1/n) > 1 \) for \( \gamma \in (\gamma_1, \gamma_2) \) where \( \gamma_1 \) and \( \gamma_2 \) are the solutions of \( G(1/n) = 1 \). Since \( G' < 0 \), there exists a \( \hat{\beta}(\gamma, n) > 1/n \) such that \( G(\hat{\beta}(\gamma, n)) = 1 \). Then relation (36) indicates that \( \hat{\beta}(\gamma, n) \leq \hat{\beta} \). Assumption \( \beta_k \geq \hat{\beta} \) (i.e., an alternative expression of \( n_k \leq \hat{n} \)) leads to \( \beta(\gamma, n) \leq \beta_k \) implying that \( G(\beta_k) \leq 1 \) or \( \pi_k^C \leq \pi_k^B \). Hence our second result on the profit difference is that

\[
\pi_k^C \leq \pi_k^B \text{ if } n_k \leq \hat{n} \leq n.
\]

In the third case in which \( \hat{n} < n_k \leq n \), we can show that \( \pi_k^C > \pi_k^B \) is possible. Due to inequality \( \hat{n} < n_k \), solving \( G(1/n_k) = 1 \) yields two distinct solutions \( \gamma_1^k \) and \( \gamma_2^k \) for which

\[
G(\tilde{\beta}(\gamma_1^k, n_k)) = G(\tilde{\beta}(\gamma_2^k, n_k)) = 1.
\]

Since \( \beta_k = 1/n_k \), we have

\[
\tilde{\beta}(\gamma_1^k, n_k) = \tilde{\beta}(\gamma_2^k, n_k) = \beta_k,
\]

which then implies that \( G(\beta_k) = 1 \) or \( \pi_k^C = \pi_k^B \). The \( \pi_k^C = \pi_k^B \) locus, which is defined in the region with \( G(1/n) \geq 1 \), passes through the two points, \( (\gamma_1^k, n_k) \) and \( (\gamma_2^k, n_k) \). As is observed in Figure 4, the maximum value of \( \tilde{\beta}(\gamma, n) \) with respect to \( \gamma \) decreases when the number of \( n \) increases. In consequence of \( \beta_k < \hat{\beta} \), we can find the threshold value \( \hat{n} \) of \( n \) such that

\[
\max_{\gamma_0 < \gamma < 0} \tilde{\beta}(\gamma, \hat{n}) = \beta_k \text{ and max } \tilde{\beta}(\gamma, n) > \beta_k \text{ for } n < \hat{n}.
\]

The last inequality implies that there are two distinct values \( \gamma_1^k \) and \( \gamma_2^k \) such that \( \tilde{\beta}(\gamma_1^k, n) = \tilde{\beta}(\gamma_2^k, n) = \beta_k \) for \( n < \hat{n} \). Hence we have

\[
\tilde{\beta}(\gamma, n) > \beta_k \text{ or } G(\tilde{\beta}(\gamma, n)) < G(\beta_k) \text{ for } \gamma \in (\gamma_1^k, \gamma_2^k) \text{ and } n < \hat{n}.
\]

Noticing \( G(\tilde{\beta}(\gamma, n)) = 1 \) leads to our third result on the profit difference,

\[
\pi_k^C > \pi_k^B \text{ if } n_k < n \leq \hat{n}
\]

and

\[
\pi_k^C < \pi_k^B \text{ if } n_k < \hat{n} < n.
\]
It is worthwhile to point out that a higher-qualified firm $k$ possibly earns more profits under Cournot competition if its net quality difference is large to the extent that $\beta_k < \beta_k(\gamma, n)$. In the case when the goods are complements, the dominance of Cournot profit over Bertrand profit is not observed in Singh and Vives (1984) and Häckner (2000). Figure 5 is an enlargement of the second quadrant of Figure 1 and represents the division of the feasible region $D^B$ when $\beta_k = 0.1$ (i.e., $n_k = 10$). Notice first that the white region is a union of the region with $1 + (n - 1)\gamma < 0$ and the region with $n < n_k$ and thus is eliminated from further considerations. The least dark-gray region is illustrated inside the unstable darker-gray region and is surrounded by the loci of $G(1/n) = 1$ and $n = n_k$. Clearly $G(1/n) > 1$ holds inside. The equal-profit locus of $\beta_k(\gamma, n) = \beta_k$ defined for $n \geq n_k$ divides the least dark-gray region into two parts and crosses the $G(1/n)$ locus at points $(\gamma_k^1, n_k)$ and $(\gamma_k^2, n_k)$. Cournot profit is larger than Bertrand profit in the horizontally-striped region surrounded by the equal-profit locus and the $n = n_k$ line while Bertrand profit is larger in the non-striped region. Needless to say, Bertrand profit is larger in any other regions.

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6. Solving $G(1/n) = 1$ and $\pi_k^C = \pi_k^B$ simultaneously yields $n_k = 10$, $\gamma_k^1 \simeq -0.11$ and $\gamma_k^2 \simeq -0.098$. 

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Figure 4. Various $\overline{\beta}(\gamma, n)$ curves against $\gamma$, given $n$. 

---

20
Next we turn our attention to the price difference. We have already seen that the Cournot price as well as the Bertrand price is positive when $\gamma < 0$ and that the neutral stability curve $\lambda^B_n = -1$ intercepts the locus of $1 + (n - 1)\gamma = 0$ at point $(-1/2, 3)$ so it divides the feasible region $D^B$ into the unstable region $R^B_u$ and the stable region $R^B_s$ as shown in the second quadrant of Figure 1. In addition to this, given $\beta_k$, the positive-sloping equal-price locus of $z^P(\gamma, n) = \beta_k$ also divides $D^B$ into two parts in a different way:

\[
R^B_+ = \{ (\gamma, n) \in D^B \mid z^P(\gamma, n) > \beta_k \}
\]

and

\[
R^B_- = \{ (\gamma, n) \in D^B \mid z^P(\gamma, n) \leq \beta_k \}.
\]

The location of the $z^P(\gamma, n) = \beta_k$ locus is sensitive to the value of $\beta_k$. Various combinations of $\gamma$ and $n$ determining the price differential and possible dynamic behavior can be conveniently classified according to the values of parameter $\beta_k$.

We start with a case in which firm $k$ is higher-qualified and the case of lower-qualified firm $k$ will be discussed later.

**Case 1.** $0 < \beta_k \leq 1$.

In this case, $R^B_+$ is empty because $z^P(\gamma, n) > 1$ for $(\gamma, n) \in D^B$. Consequently $p^C_k > p^B_k$ always in $D^B \subset R^B_+$. In Figure 5, the equal-price locus is located in the white region in which Assumption 2 is violated. $p^C_k$ is always stable while $p^B_k$ is stable in the lighter-gray region and unstable in the darker-gray region.

**Case 2.** $1 < \beta_k \leq \frac{8}{3}$.
When $\beta_k$ becomes greater than unity, the equal-price locus shifts downwards and crosses the locus of $1 + (n - 1)\gamma = 0$ at the point $(\gamma_1, n_1)$ with

$$\gamma_1 = \frac{1 - \sqrt{1 - \beta + \beta^2}}{\beta} \quad \text{and} \quad n_1 = 1 - \frac{1}{\gamma_1}.$$  

The equal-price locus divides the unstable region $R_u^B$ into two parts. In Figure 6 where $\beta_k = \frac{8}{3}$, $R_u^B \cap R_u^B$ is located above the equal-price locus and vertically-striped, $R_u^B \cap R_u^B$ is bounded by the equal-price locus and the neutral stability locus and $R_u^B \cap R_u^B$ is the lighter-gray region. The intersection moves downwards along the locus of $1 + (n - 1)\gamma = 0$ as $\beta_k$ increases from unity and arrives at the point $(-1/2, 3)$ when $\beta_k = 8/3$. Hence we have the following results concerning the price difference and the stability of the Bertrand price:

1. $p^C_k < p^B_k$ and $p^B_k$ is unstable for $(\gamma, n) \in R_u^B \cap R_u^B$,
2. $p^C_k > p^B_k$ and $p^B_k$ is unstable for $(\gamma, n) \in R_u^B \cap R_u^B$,
3. $p^C_k > p^B_k$ and $p^B_k$ is stable for $(\gamma, n) \in R_u^B \cap R_u^B$.

**Case 3.** $\frac{8}{3} < \beta_k \leq 4$.

When $\beta_k$ increases further from $8/3$, then the equal-price locus intercepts the neutral stability locus of $\lambda_{\beta}^B = -1$ from below at point $(\gamma_2, n_2)$ with

$$\gamma_2 = \frac{2(\beta - 4)}{5\beta - 8} \quad \text{and} \quad n_2 = \frac{5}{3} - \frac{2}{3\gamma_2}.$$  

7In Figure 5, we take $n = 8$ and limit the interval of $\gamma$ to $(-0.5, -0.1)$ only for the sake of graphical convenience. Changing the values of $n$ and enlarging the interval do not affect the qualitative aspects of the results.
As shown in Figure 7 where $\beta_k = 3$, the equal-price locus divides the unstable region $R^B_u$ into the vertically-striped gray region above the locus and the darker-gray region below. It also divides the stable region $R^B_s$ into two parts: the hatched region above the locus and the light-gray region below. Since it is not easy to see that the hatched region is bounded by the neutral stability locus and the equal-price locus, the lower-left part of Figure 7 is enlarged and is inserted into Figure 7. We have the following four possibilities concerning the price difference and the stability of the Bertrand price in this case:

\[(3-i) \ p^C_k < p^B_k \text{ and } p^B_k \text{ is unstable for } (\gamma, n) \in R^B_u \cap R^B_s,\]

\[(3-ii) \ p^C_k > p^B_k \text{ and } p^B_k \text{ is unstable for } (\gamma, n) \in R^B_u \cap R^B_s,\]

\[(3-iii) \ p^C_k < p^B_k \text{ and } p^B_k \text{ is stable for } (\gamma, n) \in R^B_s \cap R^B_s,\]

\[(3-iv) \ p^C_k > p^B_k \text{ and } p^B_k \text{ is stable for } (\gamma, n) \in R^B_s \cap R^B_s.\]

Notice that $\gamma_\beta$ can be defined for $\beta_k \leq 4$. This implies that the intersection moves upward along the neutral stability locus as $\beta_k$ increases further up to 4.

![Figure 7. Division of $D^B$ when $\frac{3}{4} < \beta_k < 4$](image)

**Case 4. $\beta_k > 4$.**

When $\beta_k$ becomes larger than 4, the equal-price locus is located below the $\lambda^B_u = -1$ locus. It then divides the stable region $R^B_s$ into two parts, $R^B_s \cap R^B_s$ and $R^B_s \cap R^B_s$. The former corresponds to the hatched region and the latter to the light-gray region in Figure 8 in which $\beta_k = 5$. The hatched region that appeared first in Figure 7 becomes larger with increasing value of $\beta_k$. The whole region $R^B_u$ is vertically striped, which means that the Bertrand price is larger than the Cournot price and is unstable. We have therefore the following results
concerning the price differences and the stability of the Bertrand price in this case.

\[(4-i)\ p^C_k < p^B_k \text{ and } p^B_k \text{ is unstable for } (\gamma, n) \in R^B_u,\]

\[(4-ii)\ p^C_k < p^B_k \text{ and } p^B_k \text{ is stable for } (\gamma, n) \in R^B \cap R^B,\]

\[(4-iii)\ p^C_k > p^B_k \text{ and } p^B_k \text{ is stable for } (\gamma, n) \in R^B_s \cap R^B_s.\]

Figure 8. Division of $D^B$ when $\beta_k > 4$

We are now in a position to present our results on the price differences with $\gamma < 0$. Proposition 1(ii) of Häckner (2000) deals with the case where the goods are complements and shows that lower-qualified firms charge higher prices under Bertrand competition than under Cournot competition when quality differences are large. The same result is obtained in our analysis (see (3-i), (3-iii), (4-i) and (4-ii)) when $\beta_k > 8/3$. However, (2-i) implies that large quality differences are not necessary to obtain $p^C_k < p^B_k$. It is shown there that even if the deviation of $\beta_k$ from unity is small enough, $p^C_k < p^B_k$ is still possible when the number of firms are relatively large. Furthermore, two new results are obtained in our analysis: it is shown first that the region of $(\gamma, n)$ with $p^C_k < p^B_k$ becomes larger as $\beta_k$ increases and second that $p^B_k$ can become unstable regardless of the values of $\beta_k$. We summarize these results in the following theorem:

**Theorem 5** (i) When firm $k$ is higher-qualified, then $p^C_k > p^B_k$ and $q^C_k < q^B_k$ always, whereas $\pi^C_k < \pi^B_k$ is possible for $\beta_k > 1/3$ and $\pi^C_k > \pi^B_k$ otherwise; (ii) when firm $k$ is lower-qualified, then $q^C_k < q^B_k$ and $\pi^C_k < \pi^B_k$ always whereas $p^C_k < p^B_k$ is possible.

The results obtained in Theorems 3, 4 and 5 are summarized in Table 1. The results with $^n \leq^s$ in Table 1 are obtained in the $n$-firm framework. Two of them, however, have already been exhibited by Häckner (2000): when the goods are complements lower-qualified firms charge higher prices under Bertrand competition than under Cournot competition (i.e., $p^C_k < p^B_k$ when $\gamma < 0$ and $\beta_k > 1$) in
his Proposition 1(ii) and when the goods are substitutes, higher-qualified firms earn higher profits under Bertrand competition than under Cournot competition (i.e., $\pi^C_k < \pi^B_k$ when $\gamma > 0$ and $\beta_k < 1$) in his Proposition 2(ii). We first confirm these results and then classify the parameter region into specified sub-region in which these results hold as seen in Figures 2-8 except Figure 4. In addition, we demonstrate two new results: when the goods are complements, then higher-qualified firms earn higher profits under Cournot competition (i.e., $\pi^C_k > \pi^B_k$ when $\gamma < 0$ and $\beta_k < 1$) and when the goods are substitutes, then lower-qualified firms may produce more output under Cournot competition (i.e., $q^C_k > q^B_k$).

<table>
<thead>
<tr>
<th></th>
<th>Substitutes ($\gamma &gt; 0$)</th>
<th>Complements ($\gamma &lt; 0$)</th>
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<tbody>
<tr>
<td><strong>Higher-qualified</strong> ($\beta_k &lt; 1$)</td>
<td>$p^C_k &gt; p^B_k$</td>
<td>$p^C_k &gt; p^B_k$</td>
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<tr>
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<td>$\pi^C_k \leq \pi^B_k$</td>
<td>$\pi^C_k \leq \pi^B_k$</td>
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<tr>
<td><strong>Lower-qualified</strong> ($\beta_k &gt; 1$)</td>
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<td>$\pi^C_k &gt; \pi^B_k$</td>
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Table 1. Comparison of Cournot and Bertrand strategies

4 Concluding Remarks

Singh and Vives (1984) have shown that the duopoly model with linear demand and cost functions have definitive results concerning the nature of Cournot and Bertrand competition as it was mentioned in the Introduction. Häckner (2000) increases the number of firms to $n$ from 2 and exhibits that some of these results are sensitive to the duopoly assumption. In this study, we examine the general $n$-firm oligopoly model and add two main findings to the existing literature on Cournot and Bertrand competitions. The first finding is concerned with the stability of Cournot and Bertrand equilibria. As stated in Theorems 1 and 2, Cournot equilibrium may be unstable whereas Bertrand equilibrium is always stable when the goods are substitutes. It is further shown that Bertrand equilibrium may be unstable whereas Cournot equilibrium is always stable when the goods are complements. This finding extends the stability result of Theocharis (1960) that a Cournot oligopoly model is unstable if more than three firms are involved and the goods are homogenous (i.e., perfectly substitutes).

The second finding is concerned with the comparison of Cournot and Bertrand strategies. In addition to the inequality reversal of the price and quantity differences, the profit differences shown in the duopoly framework may be reversed in the $n$-firm framework. Furthermore, as shown in Figures 2 and 5,
the horizontally-striped regions are located inside the instability regions. This means that, for example, \( C_k^c < B_k^b \) is possible when \( \gamma > 0 \) and \( \beta_k < 1 \) however \( C_k^c \) is locally unstable. The result of \( C_k^c < B_k^b \) does not have much economic implication from the dynamic point of view. Therefore the natural question to be next raised should be concerned with the global dynamic properties of the locally unstable model. Matsumoto and Szidarovszky (2010) have already started their research in this direction.

References